Alfve'n Waves in Ideal MHD Plasma

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Abstract

In MHD, the magnetic fields are frozen into the fluid and are elastic; displacing fluid elements causes magnetic restoring forces to act. This action appears as distorted magnetic field lines due to torsional and compressional Alfve'n waves. The eigenvectors are obtained giving the criteria for the Alfve'n wave's applications.

Keywords: Alfve'n wave, sound wave, magnetoacoustic wave.

الملخص :

في البلازما الهيدروديناميكية المغناطيسية المثالية ، يتم تجميد المجالات المغناطيسية في الموائع و يؤدي ذلك لاكتساب الموائع خصائص المرونة. يظهر ذلك كخطوط مجال مغناطيسي مشوهة بسبب موجات ألفاين في أشكال التوائية وتضاغطية. تم الحصول على المتجهات الذاتية مع إعطاء معايير تطبيقات موجة ألفاين.

الكلمات المفتاحية: موجة ألفين ، موجة صوتية ، موجة صوتية مغناطيسية

1. Introduction

In MHD, the magnetic fields are frozen into the fluid and are elastic; displacing fluid elements causes magnetic restoring forces to switch on. This action appears as distorted magnetic field lines due to torsional and compressional Alfve'n waves, (Davidson 2016 and Haines 2011). The field of Alfve'n waves application is vast ranging from the interstellar, the earth magnetosphere to the field of fusion plasma physics, (Hosking et al 2020). The usual one-fluid equations are normally used in plasma. The model for a magnetoplasma is given by the MHD equations, so the first aim is to give a full list of MHD equations, with the criteria of their applicability for wave propagation. The validity conditions under which the MHD equations are used require the wave frequency to be, $\omega \ll \omega_{ci}$ and the eigenvectors are obtained.

In ordinary fluid dynamics, the basic dependent variables are; the density (ρ), the temperature (T) and the velocity ($\vec{v}$) with the addition to the magnetic field induction ($\vec{B}$) for plasma. For the transport of momentum the Force law is required. This group of equations is given in section (1), they defined what is meant by 'Magnetohydrodynamic' (MHD).The simplest model for a magnetoplasma is given by the MHD equations. In section (2) the Alfve'n waves are studied and the criteria of their applicability is shown, (Nazarenko 2011 and Parra 2019).

2. Linearized MHD Equations

Assuming that equilibrium quantities $\vec{v}_0 = 0$ and $(\rho_0\ , p_0\ , \vec{B}_0\ )$ are constant in time and using the general perturbations function $\psi(\vec{\xi}, t)$ for a given mode keeping only first order terms, where, we have introduced the fluid displacement field, (Freidberg 2014) $\vec{\xi}$:

$$\psi(\vec{\xi}, t) = \psi_0 + \hat{\psi}\ exp^{i\left(\vec{k}\cdot\vec{\xi} - \omega t\right)}\ . \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}\right) p = -\gamma p \vec{\nabla} \cdot \vec{v} \tag{3}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right) \tag{4}$$

$$\rho\left(\frac{\partial v}{\partial t} + \vec{v} \cdot \vec{\nabla}\vec{v}\right) = \vec{\nabla} p + \frac{1}{4\pi}\left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} \tag{5}$$

In what follows, we shall study various solutions and asymptotic regimes of these equations.

2.1 MHD in a Straight Magnetic Field

Equations Eq.(2-5) have a very simple static, uniform equilibrium solution:

$$\rho_0 = const.,\ p_0 = const.,\ \ \vec{v}_0 = 0,\ \ \vec{B}_0 = B_0 \hat{e}_z \tag{6}$$

We shall study this case carefully, because it is very generic in the sense that it resembles many other, more complicated, equilibria situations.

2.2. MHD Waves

If we have equilibrium solution of any set of equations, our first aim ought to be to perturb it and see what happens: the system might support waves, instabilities, possibly any other interesting nonlinear behavior of small perturbations. So we seek solutions to the MHD equations Eq.(2-5) in the form of

$$\rho = \rho_0 + \hat{\rho}\,,\ p = p_0 + \hat{p},\ \vec{v} = \frac{\partial \vec{\xi}}{\partial t}\,,\ \vec{B} = B_0 \hat{e}_z + \hat{B} \tag{7}$$

To start with, we consider all perturbations to be infinitesimal and so linearize the MHD equations Eq.(2-5) as follows: using $\vec{v} = \frac{\partial \vec{\xi}}{\partial t}$ we obtain;

$$\hat{\rho} = -\vec{\nabla}\cdot(\rho_0 \vec{\xi}) \qquad \Rightarrow \frac{\hat{\rho}}{\rho_0} = -\vec{\nabla}\cdot(\vec{\xi}) \tag{8}$$

$$\hat{p} = -\vec{\xi}\cdot\vec{\nabla}p_0 - \gamma p_0 \vec{\nabla}\cdot\vec{\xi} \qquad \Rightarrow \frac{\hat{p}}{p_0} = -\gamma\vec{\nabla}\cdot\vec{\xi} \tag{9}$$

$$\hat{B} = \vec{\nabla}\times(\vec{\xi}\times\vec{B}_0) \qquad \Rightarrow \frac{\hat{B}_\perp}{\vec{B}_0} = \vec{\nabla}_\parallel \vec{\xi}_\perp\ ,\ \ \frac{\hat{B}_\parallel}{B_0} = -\vec{\nabla}_\perp\cdot\vec{\xi}_\perp \tag{10}$$

Where $\parallel$ and $\perp$ denote projections onto the direction $(\hat{e}_z)$ of $\vec{B}_0$ and onto the plane (x, y) perpendicular to it, respectively. Equations Eq.(10) tell us that parallel displacements produce no perturbation of the magnetic field, that is because the magnetic field is carried with the fluid flow and nothing will happen if a straight uniform field is displaced parallel to itself. Recall that $\hat{\rho}$, $\hat{p}$ and $\hat{B}$ are all vectors expressed as linear operators in $\vec{\xi}$.

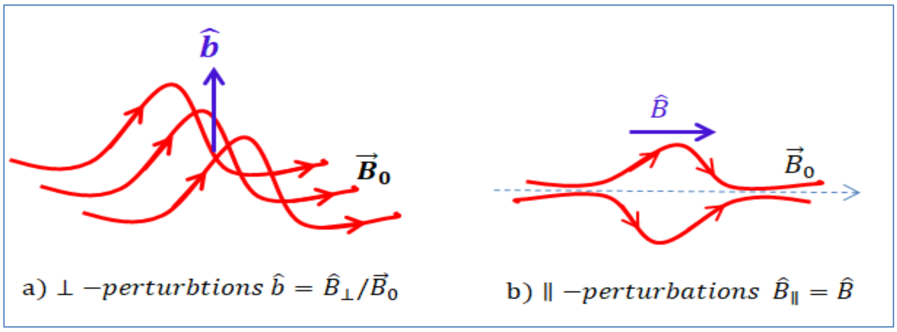

Figure (1): the perturbations of the magnetic field

The physics of magnetic-field perturbations becomes clearer if we observe that

$$\frac{\hat{B}}{\vec{B}_0} = \hat{b} + \frac{\hat{B}}{\vec{B}_0}\hat{e}_z \tag{11}$$

The perturbed field-direction vector $\hat{b}$ must be perpendicular to $\hat{e}_z$ (otherwise the field direction is unperturbed. Therefore, the perpendicular and parallel perturbations of the magnetic field are the perturbations of its direction and strength, respectively, Fig. (1):

$$\frac{\hat{B}_\perp}{\vec{B}_0} = \hat{b}, \quad \frac{\hat{B}_\parallel}{B_0} = \frac{\hat{B}}{\vec{B}_0} \tag{12}$$

Finally the force term in the momentum equation is linearized, Eq. (5):

This leads to:

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = \vec{\nabla}\left(\vec{\xi} \cdot \vec{\nabla} p_0 + \gamma p_0 \vec{\nabla} \cdot \vec{\xi}\right) + \frac{1}{c} \vec{j}_0 \times \hat{B} + \frac{1}{4\pi}\left(\vec{\nabla} \times \hat{B}\right) \times \vec{B}_0$$

$$= \gamma p_0 \vec{\nabla} \vec{\nabla} \cdot \vec{\xi} + \frac{B_0^2}{4\pi}\left(\vec{\nabla}_\perp \vec{\nabla}_\perp \cdot \vec{\xi}_\perp + \nabla_\parallel^2 \vec{\xi}_\perp\right) \tag{13}$$

Where $\vec{j}_0 = \frac{c}{4\pi}\left(\vec{\nabla} \times \vec{B}_0\right)$, we have used Eq.(2) and Eq.(3) for $\hat{\rho}$, $\hat{p}$, respectively, and used Eq.(10) and Eq.(12) into Eq.(13), where $\hat{B}$ is given as: $\hat{B} = \vec{\nabla} \times \left(\vec{\xi} \times \vec{B}_0\right)$. To obtain:

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} = c_s \vec{\nabla} \vec{\nabla} \cdot \vec{\xi} + c_a\left(\vec{\nabla}_\perp \vec{\nabla}_\perp \cdot \vec{\xi}_\perp + \nabla_\parallel^2 \vec{\xi}_\perp\right) \tag{14}$$

Where two special velocities have emerged:

$$c_s = \sqrt{\frac{\gamma p_0}{\rho_0}} \quad \text{and} \quad c_a = \frac{B_0}{\sqrt{4\pi \rho_0}} \tag{15}$$

The sound speed and the Alfve'n speed, respectively. The former is familiar from fluid dynamics, while the latter is another speed, arising in MHD, at which perturbations can travel. We shall see momentarily how this happens.

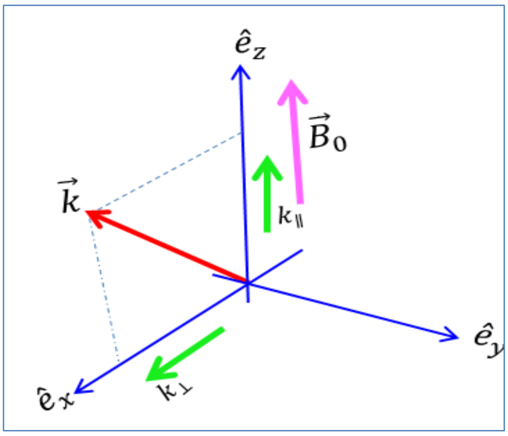

Figure (2): Coordinate system for the treatment of MHD waves.

Assume wave-like solutions of Eq. (14), $\vec{\xi} \propto exp^{i(\omega t+\vec{k}\cdot\vec{r})}$. For such perturbations,

$$\omega^2\vec{\xi} = c_s\vec{\mathrm{k}}\vec{\mathrm{k}}\cdot\vec{\xi} + c_a\left(\vec{\mathrm{k}}_\perp\vec{\mathrm{k}}_\perp\cdot\vec{\xi}_\perp + \mathrm{k}_\parallel^2\vec{\xi}_\perp\right) \tag{16}$$

Without loss of generality, let $\vec{k} = (\vec{k}_\perp, 0, \vec{k}\vec{k})$ (i.e., by definition, $\hat{e}_x$ is the direction of $\vec{k}_\perp$, see Fig.(2). Then Eq.(16) becomes,[1]:

$$\omega^2\xi_x = c_s^2 k_\perp(k_\perp\xi_x + k_\parallel\xi_\parallel) + c_a^2 k^2\xi_x \tag{17}$$

$$\omega^2\xi_y = c_a^2 k_\parallel^2\xi_y \tag{18}$$

$$\omega^2\xi_\parallel = c_s^2 k_\parallel(k_\perp\xi_x + k_\parallel\xi_\parallel) \tag{19}$$

The perturbations of the rest of the fields are

$$\frac{\hat{\rho}}{\rho_0} = -i\vec{k}\cdot\vec{\xi} = -i(k_\perp\xi_x + k_\parallel\xi_\parallel) \ , \ \frac{\hat{p}}{p_0} = \gamma\frac{\hat{\rho}}{\rho_0} \tag{20}$$

$$\hat{b} = ik_\parallel\xi_\perp = ik_\parallel\begin{pmatrix}\xi_x\\ \xi_y\\ 0\end{pmatrix} , \qquad \frac{\hat{B}}{B_0} = ik_\perp\xi_x \tag{21}$$

3. Alfve'n Waves

We observe, instantly, that Eq.(18) decouples from the rest of the system. Therefore,

$\vec{\xi} = (0, \xi_y, 0)$ Is an eigenvector, with two associated eigenvalues:

$$\omega = k_{\parallel} c_a \tag{22}$$

This is representing Alfve'n waves propagating parallel and antiparallel to $\vec{B}_0$. Alfve'nic perturbations in the fields are:

$$\vec{\xi} = \xi_y \hat{e}_y, \ \ \hat{\rho} = 0, \ \ \hat{p} = 0, \hat{B} = 0, \ \ \hat{b} = ik_{\parallel}\xi_y \hat{e}_y \tag{23}$$

i.e., it is incompressible and only involves magnetic field lines behaving as elastic strings, springing back against perturbing motions, due to the restoring curvature force, Fig. (3a).

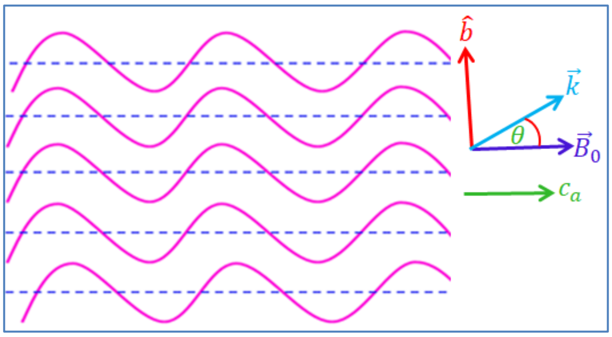

Figure (3): Distortion in the magnetic field lines due to torsional Alfve'n waves.

3.1. Magnetosonic Waves

Equations (13.12) and (13.14) form a closed 2D system:

$$\omega^2 \begin{pmatrix} \xi_x \\ \xi_{\parallel} \end{pmatrix} = \begin{pmatrix} c_s^2 k_{\perp}^2 + c_a^2 k^2 & c_s^2 k_{\parallel} k_{\perp} \\ c_s^2 k_{\parallel} k_{\perp} & c_s^2 k_{\parallel}^2 \end{pmatrix} \begin{pmatrix} \xi_x \\ \xi_{\parallel} \end{pmatrix} \tag{24}$$

The resulting dispersion relation is

$$\omega^4 - k^2 (c_s^2 + c_a^2) \omega^2 + c_s^2 c_a^2 k^2 k_{\parallel}^2 = 0 \tag{25}$$

The four roots of this equation are:

$$\omega^2 = \frac{1}{2}k^3\left\{c_s^2 + c_a^2 \pm ((c_s^2 + c_a^2)^2 - 4c_s^2 c_a^2 cos\theta^2)^{\frac{1}{2}}\right\}, \quad cos\theta \equiv \frac{k_\parallel}{k} \tag{26}$$

The two "+" solutions are the "fast magnetosonic waves" and the two "–" ones are the "slow magnetosonic waves". Since both sound and Alfve'n speeds are involved, it is obvious that the key parameter demarcating different physical regimes will be their ratio, or, conventionally, the ratio of the thermal to magnetic energies in the MHD medium, known as the plasma beta:

$$\beta \equiv 8\pi \frac{p_0}{B_0^2} = \frac{2}{\gamma}\frac{c_s^2}{c_a^2} \tag{27}$$

Clearly, magnetosonic wave contain perturbations of both the magnetic field and of the hydrodynamic quantities $\rho, p, \vec{v}$, but working them all out for the case of general oblique propagation is involved. The physics of what is going on is best understood via a few particular case, (Kunz et al 2020):

3.2. Parallel Propagation

Consider the orientation $k_\perp = 0\,, (\theta = 0)$. Then $(\xi_x, 0, 0)$ and $(0, 0, \xi_\parallel)$ are eigenvectors of the matrix in Eq.(24) and the two corresponding waves are:

a) An Alfve'n wave, with perturbation in the $\hat{e}_x$ −direction:

$$\omega^2 \xi_x = k_\parallel^2 c_a^2 \xi_x \implies \omega = \pm k_\parallel c_a \tag{28}$$

$$\vec{\xi} = \xi_x \hat{e}_x, \quad \hat{\rho} = \hat{p} = \hat{B} = 0, \quad \hat{b} = ik_\parallel \xi_x \hat{e}_x \tag{29}$$

At high β, it becomes the slow wave, at low β, it becomes the fast wave.

b) The parallel-propagating sound wave:

$$\omega^2 \xi_\parallel = k_\parallel^2 c_s^2 \xi_\parallel \Rightarrow \omega = \pm k_\parallel c_s \tag{30}$$

$$\vec{\xi} = \xi_\parallel \hat{e}_z \ \ , \frac{\hat{\rho}}{\rho_0} = ik_\parallel \xi_\parallel \,, \ \frac{\hat{p}}{p_0} = \gamma \frac{\hat{\rho}}{\rho_0}\,, \ \ \hat{B} = \hat{b} = 0 \tag{31}$$

At high β, this becomes the fast wave, at low β, it changes into the slow wave, the magnetic field does not participate at all in this case.

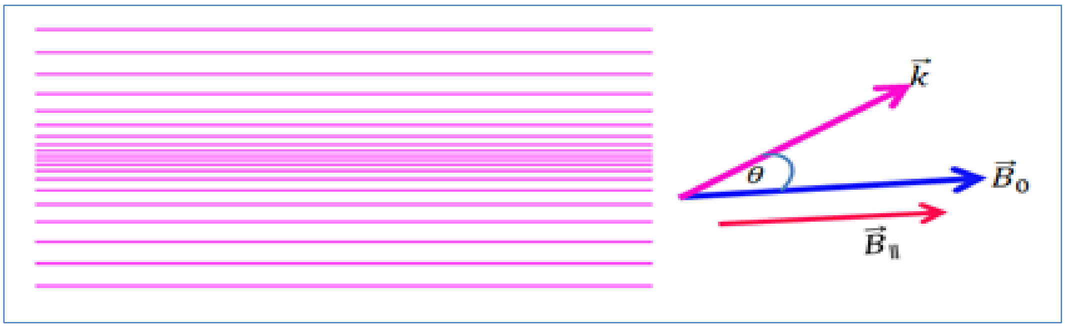

Figure (4): Compression of the magnetic field lines due to compressional Alfve'n waves.

3.3. Perpendicular Propagation

Now consider the orientation, $k_{\parallel} = 0$, $(\theta = \frac{\pi}{2})$. Then $(\xi_x, 0, 0)$ is again an eigenvector of the matrix in Eq.(24). The resulting fast magnetosonic wave is again a sound wave, but because it is perpendicularly propagating, both thermal and magnetic pressures get involved. The perturbations are compressions and rarefactions in both the fluid and the field, and the speed at which they travel is a combination of the sound and Alfve'n speeds, (Kunz et al 2020), with the latter now representing the magnetic pressure response:

$$\omega^2 \xi_x = k_{\perp}^2 (c_s^2 + c_a^2) \xi_x \implies \omega = \pm k_{\perp} \sqrt{c_s^2 + c_a^2} \tag{32}$$

$$\vec{\xi} = \xi_x \hat{e}_x \,, \quad \frac{\hat{\rho}}{\rho_0} = -ik_{\perp}\xi_x \,, \frac{\hat{p}}{p_0} = \gamma \frac{\hat{\rho}}{\rho_0} \,, \frac{\hat{B}}{B_0} = -ik_{\perp}\xi_x \,, \hat{b} = 0 \tag{33}$$

Note that the thermal and magnetic compressions are in phase and there is no bending of the magnetic field, Fig.(4a).

4. Conclusion

It is interesting to notice that the associated magnetic perturbation $\hat{b} = ik_{\parallel}\xi_y \hat{e}_y$ in Eq.(23) acts perpendicularly to the confining magnetic field $\vec{B}_0 = B_0 \hat{e}_z$. Therefore, it induces torsion of the magnetic field lines and is called torsional Alfve'n wave (see Fig. (3a)).

Note from Eq. (33) that the thermal $\left(\frac{\hat{p}}{p_0} = -i\gamma k_{\perp}\xi_x\right)$ and magnetic compressions $\left(\frac{\hat{B}}{B_0} = -ik_{\perp}\xi_x\right)$ are in phase and there is no bending of the magnetic field, (see Fig. (4a)), this mode is therefore commonly referred to as the compressional Alfve'n wave. Also remember that there are two types of perturbations involve, in the strength and in the direction of the magnetic field.

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