COUNTING THE ELEMENTS OF BELLISSIMA'S CONSTRUCTION FOR FREE HEYTING ALGEBRA

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Abstract

In this paper, we successfully have counted the elements of Bellissima's construction of two generators of level two manually by using combinations and found that they are 265428 elements; essentially we have built a python code to count the numbers of the elements of Bellissima's construction.

Keywords: intuitionistic logic, Generic Kripke models, Free Heyting Algebra, Upward closed sets.

الملخص

هذه الورقة تحتوي على طريقتين لحساب عدد عناصر مستويات تركيبة بيليسما لجبر هايتنق الحر المولد من عنصرين، يدويا باستخدام التوافيق والتباديل وباستخدام برنامج حاسوبي بلغة البايثون وتمت مطابقة النتائج في الحالتين.

1. Introduction:

The relationship that exists between Heyting algebras and intuitionistic logic can be compared to the association that is found between Boolean algebras and classical logic... Both types of algebras can be characterized as distributive lattices, and each constitutes a distinct variety. A notable distinction lies in the fact that the free Heyting algebras generated are infinite in nature, whereas Boolean algebras are characterized by finiteness. Nonetheless, Heyting algebras embody a concept analogous to that of Boolean algebras; this was elucidated by Mckinsey and Tarski in their seminal publication on Heyting algebras in the 1940s.

Nevertheless, among the free Heyting algebras examined thus far, the sole case of a single generator has been thoroughly elucidated through the application of the Rieger-

Nishimura ladder; however, for algebras with two or more generators, the underlying structure remains enigmatic, despite the existence of various known properties. Bellissima effectively represented the finitely generated free Heyting algebras utilizing Kripke models for this purpose. Fundamentally, Grigolia and Esakia proposed an analogous construction.

To the best of our knowledge that the number of the elements up to level one of these construction is unknown.

Therefore, our interest was to count the elements in the levels of Bellissima's construction.

2. Some of the basic definitions

Definition 2.1 (Sankappanavar & Burris, 1981)

Heyting algebras represent the algebraic models for intuitionistic logic. They are defined as an Algebra $A = (A,\vee,\wedge,\rightarrow,0,1)$ that is a bounded distributive lattice with least elements 0 and greatest element 1, and for all $w \in$ A, $w \rightarrow v$ is the greatest element of z of A such that $w \wedge z \preccurlyeq v$, where this is called a pseudo-complement of x with respect to y, the operation $\rightarrow$ is called Heyting implication and $\preccurlyeq$ is defined by $x \preccurlyeq y$ if and only if $w \wedge v = w$.

In addition, Heyting algebra can be defined geometrically as an Algebra $A = (A,\vee,\wedge,\rightarrow,0,1)$ with two nullary and three binary operations if it satisfies the following:

$A1: \langle A,\vee,\wedge\rangle$ *is a distributive lattice*

$A2: x \wedge 0 = 0; w \vee 1 = 1$

$A3: x \rightarrow x = 1$

$A4: x \wedge (w \rightarrow v) = w \wedge v;\ (w \rightarrow v) \wedge v = v$

$A5: x \rightarrow (v \wedge z) = (w \rightarrow z) \wedge (w \rightarrow v)$

$(v \vee z) \rightarrow z = (w \rightarrow z) \wedge (v \rightarrow z)$

Examples

1. Each Boolean algebra can be defined as a Heyting algebra supplemented with $p \longrightarrow q$ given by $\neg p \vee q$.
2. A topological space (X,τ) , where O(X) is the family of all open sets in X provides a complete Heyting algebra for $w, v \in \tau$ we define

$w \cap v = w \wedge v, w \cup v = w \vee v,\ x \sqsubseteq v \Leftrightarrow w \subseteq v,\ X = 1, \emptyset = 0$ and

$w \rightarrow v = \overline{w - v}^{c} = (w^{c} \cup v)^{o}$

(Where c is the complement, $\overline{}$ topological closure and o the interior.)

Such algebras are called topological Heyting algebras and are considered one of the most important examples of them.

3. The propositional formulae in κ propositional variables in the intuitionistic propositional calculus up to equivalence, together they form a Heyting algebra named $IPL\kappa$.This algebra is freely generated by the (equivalence classes of the) propositional variables, therefor it is isomorphic to the free Heyting algebra $F\kappa$ over κ generators.
4. If (X,≼) is a partial ordering, then the decreasing sets of the partial ordering which are the closed sets of $\mathcal{O}\uparrow(X,\preccurlyeq)$ form a topology and hence a Heyting algebra $\mathcal{O}\downarrow(X,\preccurlyeq)$, and the sets increasing form a topology as well and hence a Heyting algebra$\mathcal{O}\uparrow(X,\preccurlyeq)$, Therefor such an algebra is bi-Heyting, and the two lows of infinite distributive can be proved.

Definition 2.2

Kripke models: they are considered to be an important model of intuitionistic logic and are defined as a $K = \langle K, \preccurlyeq, \alpha\rangle$, where K is a non-empty set, $\preccurlyeq$ is a partial ordering of K and α is a mapping from the collection that encompasses all propositional variables relevant to intuitionistic propositional logic to a specific element within the power set of K, whereby for every propositional variable p and $w, v \in K$ we have

If w ∈ $\alpha(p)$ and w ≼ v, then v ∈ $\alpha(p)$, calling this property by the monotonicity property.

The mapping α from the set of all propositional formulas can be extend to a valuation ρ to the power set of in the subsequent manner:

For any propositional variable

1) $(p) = (p)$, where p is any propositional variable.
2) $\rho(\perp) = \emptyset$
3) $\rho(\phi \wedge \psi) = \rho(\phi) \cap \rho(\psi)$
4) $\rho(\phi \vee \psi) = \rho(\phi) \cup \rho(\psi)$
5) $\rho(\phi \to \psi) = \{z \in K | \{w\} \uparrow \cap\ \rho(\phi) \subseteq \rho(\psi)\}$.

By considering a set of formulas such that $\Gamma = \{\phi_1, \dots, \phi_n\}$, then the valuation Γ is equivalent to the valuation of the conjunction among all formulas within the set Γ, i.e.

6) $\rho(\Gamma) = \rho(\phi_1 \wedge \dots \wedge \phi_n) = \rho(\phi_1) \cap \dots \cap \rho(\phi_n)$
7) If Γ is empty set of formulas then $\rho(\Gamma) = K$.

Definition 2.3 (Fitting & Mendelsohn ,2012).

Let K be an intuitionistic Kripke model. The relation (denoted $K, w \vDash \phi$) is the relation between elements of K and propositional formulas and we say that w forces or satisfies ϕ when it is defined as follows:

w $\in \rho(\phi)$ if and only if $K, w \vDash \phi$, for any w $\in K$ and any formula ϕ.

Now the explanation of how intuitionistic Kripke models represents intuitionism.

The different stages of information are represented by the elements of the set K such that any w in K is a known fact at a particular time. The partial order $\preccurlyeq$ signifies the progressive phases that are attained through the acquisition of additional information. In other words, if we consider two stages of information, denoted as w and v such that w $\preccurlyeq$ v, this implies that the information contained in w is also encompassed in v, along with potentially additional information. If we have a particular fact, the forcing relation $\vDash$ delineates the formulas that can be inferred to be true.

Definition 2.4 (Elageili & Truss , 2012)

K be a Kripke model, then

1) If a point w in K has a forcing relation with a formula ϕ ($w \vDash \phi$) then it is said that a formula ϕ is valid at a point w also we mention that ϕ is valid in K, written as $K \vDash \phi$, if $w \vDash \phi$ for all a in K.
2) A set of formulas Γ is deemed to be valid at a point w in K if $w \vDash \phi$ for all $\phi \in \Gamma$.

Additionally, if K is a theory we mention that Γ is valid in K, denoted as $K \vDash \Gamma$, if Γ is valid at each point of K.

3) A formula ϕ is deemed to be a Kripke consequence of a set of formulas Γ, denoted as $\Gamma \vDash \phi$, if ϕ is valid in K whenever Γ is valid in K.
4) Moreover, A formula ϕ is called a Kripke valid if $\emptyset \vDash \phi$ or in essence we denote it as $\vDash \phi$.

3. Applications of Kripke models in the realm of intuitionistic propositional logic

i) Counter models

In intuitionistic propositional logic it has been seen that a formula ϕ is provable if and only it is valid in each Kripky model, wherefore a formula ϕ in intuitionistic propositional logic is unprovable if found a finite Kripke model K such that ϕ is invalid in it, this means , $\rho(\phi) \neq K$. These models are called ***counter models.***

ii) Generic Kripke models

The generic or universal model refers to Kripke models that are designed to exemplify the Heyting algebra, which integrates intuitionistic propositional expressions across n variables. Subsequently, we will demonstrate that this Heyting algebra exhibits an isomorphic relationship with the unrestricted Heyting algebra. $F_n(R)$ of n generators, which possess the characteristic of universal mapping applicable to the category of Heyting algebras. The generic Kripke model are defined by following the same process in (Darniere & Junker, 2010), (Bellissima, 1986), (Elageili & Truss,2012). The generic Kripke model R_n=$\langle K_n, \preccurlyeq_n, \rho_n \rangle$, is construct throughout a chain of Kripke models R_n^d=$\langle K_n^d, \preccurlyeq_n^d, \rho_n^d \rangle$ ordered by incorporation In such a manner that each successive model is derived from its predecessor through the incorporation of an additional layer beneath. It is imperative that each of these models be diminished with the understanding that there exist no two discrete points w,v with a matching valuation so that v functions as the only cover for w, or, in such a manner that every element that strictly dominates w also strictly dominates v. For both cases w and , the same formulas are satisfied, so the theory of the model is not effected if we can ignore one of them.

Through engagement with intuitionistic propositional logic, Bellissima employed a Kripke model $K = \langle K, \preccurlyeq, \alpha \rangle$ from the components of K to the power set of all formulas and established the valuation ρ on it.

Therefore, we can define the forcing relation ⊨ by w ⊨ ϕ ⟺ ϕ ∈ ρ (w), for every w ∈ K and for any formula ϕ.

Definition 4.1 (Shabana & Elageili, 2018**)**

A set X is called *upward-closed* if for any $x \in$, we have

$$\{x\} \uparrow = \{y \in X : y \succcurlyeq x\} \subseteq X$$

Now, we will clarify a method for creat R_n.

Let the set $P_n = \{p_i : i < n\}, 0 \leq n < \omega$ be in intuitionistic propositional variables, and then we denote the set of the valuation as

$$val_n = \{\beta : \beta \subseteq P_n\} = 2^{Pn}$$

The model R_n^d is defined by induction on d (where d is the level and n is the number of generators) as explained in the next steps:

1) Construct the set of cases K_n^d. First, by denote $K_n^{-1} = \emptyset$. Hence the elements $w_{\beta,Y}$ of the level $K_n^d \setminus K_n^{d-1}$ satisfy the next conditions:

i. Y is an upward-closed set within the preceding model and is required to intersect with the terminal level of it. Consequently, for d=0, we obtain Y=$\emptyset$.

ii. $\beta \in val_n$ such that $\beta \subseteq \bigcap_{w \in Y} \rho_n^{d-1}(w)$. Thus, if d=0, then the number of elements in K_n^0 is equal to the number of elements in the power set of the propositional variables val_n.

iii. If Y is an upward-closed for some element $w \in K_n^{d-1}$, then we must have $\rho_n^d(w_{\beta,Y}) \subsetneq \rho_n^{d-1}(w)$. Hence, if $\rho_n^{1-d}(w)$ =$\emptyset$, then there is no element $w_{\beta,Y}$ such that $\rho_n^d(w_{\beta,Y}) \subset \emptyset$. This means that in the new level we cannot add any new element $w_{\beta,Y}$ under w.

2) Each element $w_{\beta,Y}$ can be evaluated by

$$\rho_n^d(w_{\beta,Y}) = \beta$$

3) The partial ordering $\preccurlyeq_n^{d-1}$ is extended to $\preccurlyeq_n^d$ as follows:

$$\preccurlyeq_n^d = \preccurlyeq_n^{d-1} \cup \{(w_{\beta,Y}, w) | w_{\beta,Y} \in K_n^d \setminus K_n^{d-1} \; and \; w = w_{\beta,Y} or \; w \in Y\}$$

Finally, our generic model R_n is defined by

$$K_n = \bigcup_{d<\omega} K_n^d , \preccurlyeq_n = \bigcup_{d<\omega} \preccurlyeq_n^d , \rho_n = \bigcup_{d<\omega} \rho_n^d$$

Remark 4.1

There are two conditions must be consider while constructing the elements of each level;

(i) $\beta \subseteq \bigcap_{w \in Y} \rho_n^{d-1}(w)$

(ii) If =$\{w\}\uparrow$ for some $w \in K_n^{d-1}$, we must have

$$\rho_n^d(w_{\beta,Y}) \subsetneq \rho_n^{d-1}(w)$$

The extension of ρ to the new model is valid and ensured by the first condition.

In addition, if we assume that $\rho_n^d(w_{\beta,Y}) = \rho_n^{d-1}(w)$ in the condition (ii), then the valuation of $w_{\beta,Y}$ and are same valuation and w is the unique cover of $w_{\beta,Y}$, This suggests that our model will not experience any reduction in efficacy

5. Bellissima's construction for one generator

To construct R_1 which is shown in Figure (1).

We define $P_1 = \{p\}$ $so \; val_1 = \{\emptyset, \{p\}\}$.

Now by induction on d, systematically we can show how to construct each model R_1^d.

Starting by d=0, then Y=$\emptyset$ is the only upward-closed set in R_1^{-1}, so

$$K_1^0 = K_1^0 \setminus K_1^{-1} = \{w_{\{p\},\emptyset}, w_{\emptyset,\emptyset}\} = \{w_0, w_1\} ,$$

as well,

$$\preccurlyeq_1^0 = \{(w_0, w_0), (w_1, w_1)\}$$

$$\rho_1^0(w_0) = \{p\}, \rho_1^0(w_1) = \emptyset$$

Then for $d = 1$, in R_1^0 ,

$$\{w_0\}\uparrow, \{w_1\}\uparrow, K_1^0$$

are the upward-closed subsets.

Since $\rho_1^0(w_1) = \emptyset$, then there are no new added elements under w_1. So the only desired upward-closed subsets of $R_1^0$are $\{w_0\}\uparrow, K_1^0$.

$$K_1^1 \setminus K_1^0 = \{w_{\emptyset,\{w_0\}\uparrow}, w_{\emptyset,K_1^0}\} = \{w_2, w_3\}$$

$$K_1^1 = K_1^0 \cup \{w_2, w_3\} = \{w_0, w_1, w_2, w_3\}$$

$$\preccurlyeq_1^1 = \preccurlyeq_1^0 \cup \{(w_2, w_2), (w_3, w_3), (w_2, w_0), (w_3, w_0), (w_3, w_1)\}$$

$$\rho_1^1(w_2) = \rho_1^1(w_3) = \emptyset .$$

For d=2,

$$\{w_2\}\uparrow, \{w_3\}\uparrow, \{w_1\}\uparrow \cup \{w_2\}\uparrow, K_1^1,$$

are the upward-closed subsets in R_1^1 , which intersection with the last level of the model.

There cannot be any elements added under $\{w_2\}\uparrow$, $\{w_3\}\uparrow$ due to the fact their valuation is $\emptyset$. Thus in R_1^1 , the upward-closed subsets are required that

$$\{w_1\}\uparrow \cup \{w_2\}\uparrow, K_1^1.$$

Furthermore,

$$K_1^2 \setminus K_1^1 = \{w_{\emptyset w1\}\uparrow \cup \{w2\}\uparrow,}, w_{\emptyset,K_1^1}\} = \{w_4, w_5\}$$

$$K_1^2 = K_1^1 \cup \{w_4, w_5\} = \{w_0, w_1, w_2, w_3, w_4, w_5\}$$

$$\preccurlyeq_1^2 = \preccurlyeq_1^1 = \cup \{(w_4, w_4), (w_5, w_5), (w_4, w_0), (w_5, w_0), (w_4, w_1),$$

$$(w_5, w_2), (w_5, w_3)\}$$

$$\rho_1^2(w_4) = \rho_1^2(w_5) = \emptyset.$$

Continuing like this we obtain

$$K_1 = \bigcup_{d<\omega} K_1^d, \preccurlyeq_1 = \bigcup_{d<\omega} \preccurlyeq_1^d, \rho_1 = \bigcup_{d<\omega} \rho_1^d$$

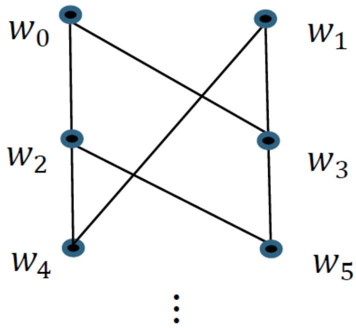

Figure (1) The generic Kripke model R_1

Definition 5.1

If each appearing letter in a formula φ belongs to P_n then φ is called an *n-formula*.

Theorem 5.1 (Bellissima, 1986)

For all $n < \omega$, and each $n - formula\ \varphi$, we have that $R_n \vDash \varphi$ if and only if φ is an intuitionistic tautologies.

Definition 5.2

The Heyting algebra associated with intuitionistic propositional formulas in n variables and for $n < \omega$ is

$A_n = (A_n, \vee, \wedge, \rightarrow, 0, 1)$ as follows :

$$A_n = \{A: A\ is\ an\ upwards\ closed\ set\ of\ R_n\}$$

$$A \vee B = A \cup B$$

$$A \wedge B = A \cap B$$

$$A \rightarrow B = \{K_n: \{x\} \uparrow \cap A \subseteq B\}$$

$$0 = \emptyset,\ 1 = K_n.$$

Now will show the Heyting algebra of intuitionistic propositional formulas in one generator p.

The upward-closed subsets in $\mathrm{R}_1$are:

$$\{w_0\} \uparrow = w_0,\ \{w_1\} \uparrow = w_1,\ \{w_2\} \uparrow = \{w_0, w_2\},$$

$$\{w_3\} \uparrow = \{w_0, w_1, w_3\},\ \{w_0\} \uparrow \cup \{w_1\} \uparrow = \{w_0, w_1\},$$

$$\{w_1\} \uparrow \cup \{w_2\} \uparrow = \{w_0, w_1, w_2\},$$

$$\{w_2\} \uparrow \cup \{w_3\} \uparrow = \{w_0, w_1, w_2, w_3\},$$

$$\{w_4\} \uparrow = \{w_0, w_1, w_2, w_4\}$$

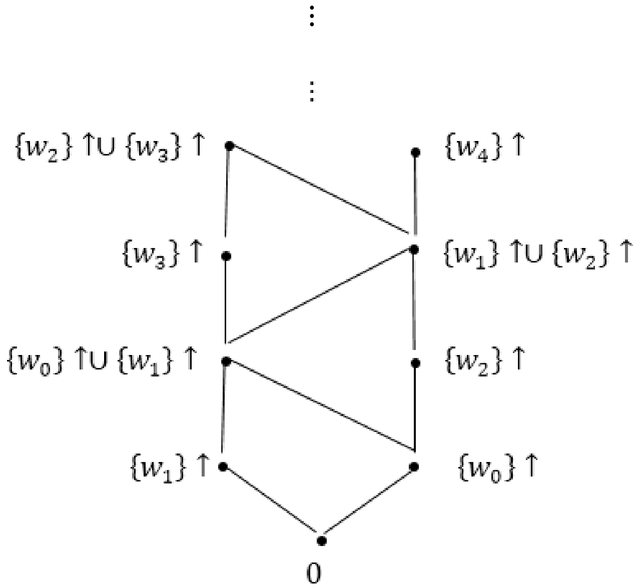

Figure (2) The Riger-Nishumera Ladder

Theorem 5.2 (Shabana & Elageili, 2018)

Let A_n be the Heyting algebra which is isomorphic to the free Heyting algebra $F_A(n)$ on n generators.

Thus, for one generator p the free Heyting algebra $F_A(1)$, where

$$F_A(1) = \{p, \neg\, p, \perp, p \vee \neg p, \neg p \to p, \dots\}$$

The operations $\vee, \wedge$ and $\to$ are the logical connectives. The contradiction statement and the tautology statement are the smallest and the largest elements in $F_A(1)$ respectively.

If we denote a map $f: F_A(1) \to R_1$ by

$$f(\psi) = \{w \in K_1 : w \vDash \psi\}.$$

Then, we get

$$f(\perp) = \emptyset$$

$$f(p) = \{w_0\} = \{w_0\}\uparrow$$

$$f(\neg p) = f(p \to \perp) = f(p) \Rightarrow f(\perp) = \{w_0\}\uparrow \Rightarrow \emptyset = \{w_1\}\uparrow$$

$$f(p \vee \neg p) = f(p) \cup f(\neg p) = \{w_0\}\uparrow \cup \{w_1\}\uparrow$$

$$f(, \neg p \to p) = f(\neg p) \Rightarrow f(p) = \{w_1\}\uparrow \Rightarrow \{w_0\}\uparrow = \{w_2\}\uparrow$$

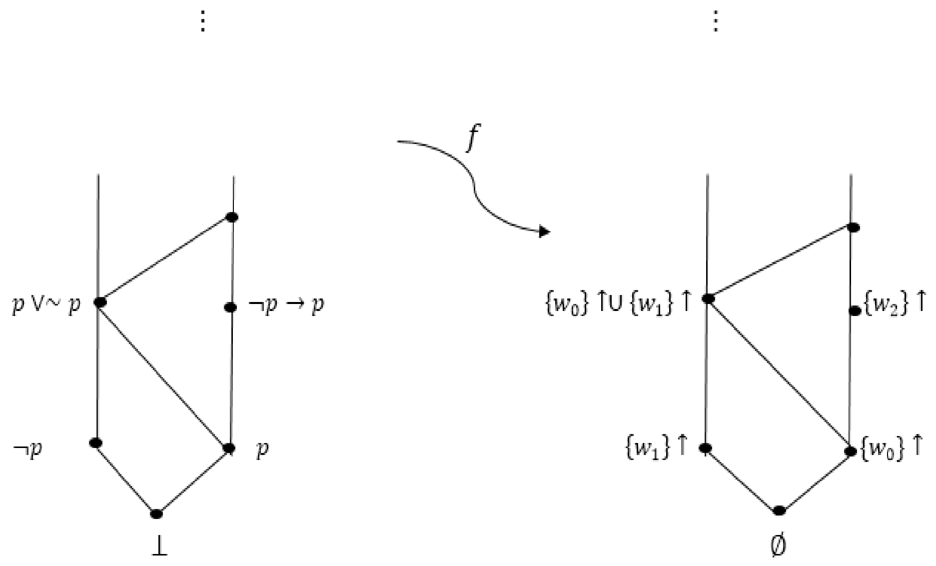

Figure (3) The isomorphism between $F_A(1)$ and A_1

6. Bellissima's construction for two generators

Let $P_2 = \{p, q\}$ be the set of intuitionistic propositional variables.

So $val_2 = \{\emptyset, \{p\}, \{q\}, \{p, q\}\}$

We systematically elucidate the methodology for the construction of each model R_2^d by induction

For $d = 0, Y = \emptyset$ is the only upward-closed set in R_2^{-1}, so

$$K_2^0 = K_2^0 \backslash K_2^{-1} = \{w_{\emptyset,\emptyset,} w_{\{p\},\emptyset}, w_{\{q\},\emptyset}, w_{\{p,q\},\emptyset}\}$$

$$= \{w_0, w_1, w_2, w_3\}$$

Also

$$\preccurlyeq_2^0 = \{(w_0, w_0), (w_1, w_1), (w_2, w_2), (w_3, w_3)\}$$

$$\rho_2^0(w_0)=\emptyset \text{ , } \rho_2^0(w_1)=\{p\} \ \rho_2^0(w_2)=\{q\}, \rho_2^0(w_3)=\{p,q\}$$

For d=1 the upward-closed subset in R_2^0 are

$\{w_0\}\uparrow, \{w_1\}\uparrow, \{w_2\}\uparrow, \{w_3\}\uparrow, \{w_0, w_1\}\uparrow, \{w_0, w_2\}\uparrow, \{w_0, w_3\}\uparrow, \{w_0, w_2\}\uparrow,$
$\{w_1, w_3\}\uparrow, \{w_2, w_3\}\uparrow, \{w_0, w_1, w_2\}\uparrow, \{w_0, w_1, w_3\}\uparrow, \{w_0, w_2, w_3\}\uparrow,$
$\{w_1, w_2, w_3\}\uparrow$, K_2^0 .

Since $\rho_2^0(w_0)=\emptyset$, under w_0 we cannot add any new element, so the required upward-closed subsets of R_2^0 are $\{w_0\}\uparrow$, $\{w_1\}\uparrow$, $\{w_2\}\uparrow$, $\{w_3\}\uparrow$, $\{w_0, w_1\}\uparrow$, $\{w_0, w_2\}\uparrow$, $\{w_0, w_3\}\uparrow$, $\{w_1, w_2\}\uparrow$, $\{w_1, w_3\}\uparrow$, $\{w_2, w_3\}\uparrow$, $\{w_0, w_1, w_2\}\uparrow$, $\{w_0, w_1, w_3\}\uparrow$, $\{w_0, w_2, w_3\}\uparrow$, $\{w_1, w_2, w_3\}\uparrow$, K_2^0.

$K_2^1 \backslash K_2^0 = \{w_{\emptyset,\{w_1\}\uparrow}, w_{\emptyset,\{w_2\}\uparrow}, w_{\emptyset,\{w_3\}\uparrow}, w_{\emptyset,\{w_0,w_1\}\uparrow}, w_{\emptyset,\{w_0,w_2\}\uparrow}, w_{\emptyset,\{w_0,w_3\}\uparrow}, w_{\emptyset,\{w_1,w_2\}\uparrow}$
$, w_{\emptyset,\{w_1,w_3\}\uparrow}, w_{\emptyset,\{w_2,w_3\}\uparrow}, w_{\emptyset,\{w_0,w_1,w_2\}\uparrow}, w_{\emptyset,\{w_0,w_1,w_3\}\uparrow}, w_{\emptyset,\{w_0,w_2,w_3\}\uparrow}, w_{\emptyset,\{w_1,w_2,w_3\}\uparrow}$
$, w_{\emptyset,K_2^0}, w_{\{p\},\{w_3\}\uparrow}, w_{\{p\},\{w_1,w_3\}\uparrow}, w_{\{q\},\{w_3\}\uparrow}, w_{\{q\},\{w_2,w_3\}\uparrow}\}$
$=\{w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}, w_{18}, w_{19}, w_{20}, w_{21}\}$
$K_2^1 = K_2^0 \cup \{w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}, w_{18}, w_{19}, w_{20}, w_{21}\}$
$\preccurlyeq_2^1 = \preccurlyeq_2^0 \cup \{(w_4, w_4), (w_5, w_5), (w_6, w_6), (w_7, w_7,), (w_8, w_8), (w_9, w_9),$
$(w_{10}, w_{10}), (w_{11}, w_{11}), (w_{12}, w_{12}), (w_{13}, w_{13}), (w_{14}, w_{14}), (w_{15}, w_{15}), (w_{16}, w_{16}),$
$(w_{17}, w_{17}), (w_{18}, w_{18}), (w_{19}, w_{19}), (w_{20}, w_{20}), (w_{21}, w_{21}), (w_4, w_1), (w_5, w_2),$

$(w_6, w_3), (w_7,\{w_0, w_1\}), (w_8,\{w_0, w_2\}), (w_9,\{w_0, w_3\}), (w_{10},\{w_1, w_2\}), (w_{11},\{w_1, w_3\}),$
$(w_{12},\{w_2, w_3\}), (w_{13},\{w_0, w_1, w_2\}), (w_{14},\{w_0, w_1, w_3\}), (w_{15},\{w_0, w_2, w_3\}),$
$(w_{16},\{w_1, w_2, w_3\}), (w_{17},\{w_0, w_1, w_2, w_3\}), (w_{18}, w_3), (w_{19},\{w_1, w_3\}), (w_{20}, w_3\}), (w_{21}$
$,\{w_2, w_3\})\}.$
$\rho_2^1(w_4) = \emptyset,\ \rho_2^1(w_5) = \emptyset, \rho_2^1(w_6) = \emptyset, \rho_2^1(w_7,) = \emptyset, \rho_2^1(w_8) = \emptyset, \rho_2^1(w_9) = \emptyset$
$\rho_2^1(w_{10}) = \emptyset,\ \rho_2^1(w_{11}) = \emptyset, \rho_2^1(w_{12}) = \emptyset, \rho_2^1(w_{13}) = \emptyset,\ \rho_2^1(w_{14}) = \emptyset, \rho_2^1(w_{15}) = \emptyset$
$\rho_2^1(w_{16}) = \emptyset, \rho_2^1(w_{17}) = \emptyset, \rho_2^1(w_{18}) = \{p\}, \rho_2^1(w_{19}) = \{p\}, \rho_2^1(w_{20}) = \{q\}, \rho_2^1(w_{21}) = \{q\}.$
Continuing like this obtain

$$K_2 = \bigcup_{d<\omega} K_2^d, \preccurlyeq_2 = \bigcup_{d<\omega} \preccurlyeq_2^d, \rho_2 = \bigcup_{d<\omega} \rho_2^d$$

The figure below shows that we have five single-parent children, $\{w_4, w_5, w_6, w_{18}, w_{20}\}$ in level one, the first three are labelled by $\emptyset$ and the other two are labelled by p and q respectively. We have two-parent children $\{w_7, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{19}, w_{20}\}$ which are w_{11} and w_{19}, that are totally covered by the set $\{w_1, w_3\}$, and the elements w_{12} and w_{20} are totally covered by the set $\{w_2, w_3\}$.It shows that we have three-parent children $\{w_{13}, w_{14}, w_{15}, w_{16}\}$ that are all labelled by $\emptyset$ and finally one element of a four-parent child w_{17}, the total number of elements in this level is eighteen

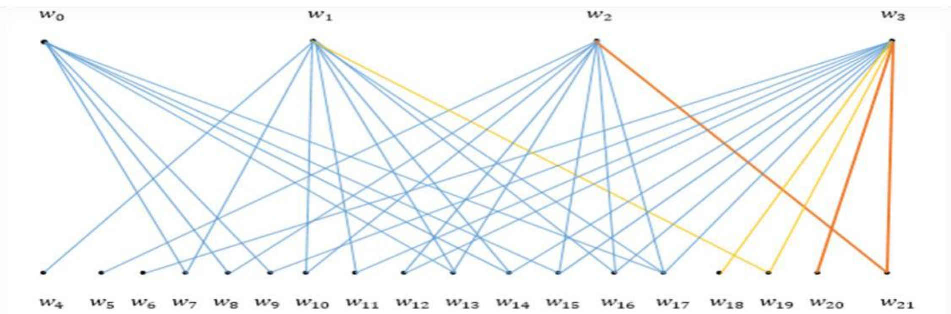

Figure (4) The generic Kripke model R_2

As for level two: K_2^2 contains 2^{18} elements and by excluding fifteen points from them, fourteen by following the conditions of the structure for the what are labelled by $\emptyset$ and $\emptyset$ itself. Therefore, the elements are 262129 generated by level one only.

Some examples of the elements in this level:

The elements generated by w_4 and another single-parent child of level one makes 4 new labels of $\emptyset$. w_4 with some of the two-parent children makes four other elements labeled by $\emptyset$. w_4 and a three-parent child w_{15} they generate another label of $\emptyset$, w_4 generates with two elements, the first is a single-parent child and the second is a two-parent child they make four new elements labeled by $\emptyset$, these are among the 262131 elements.

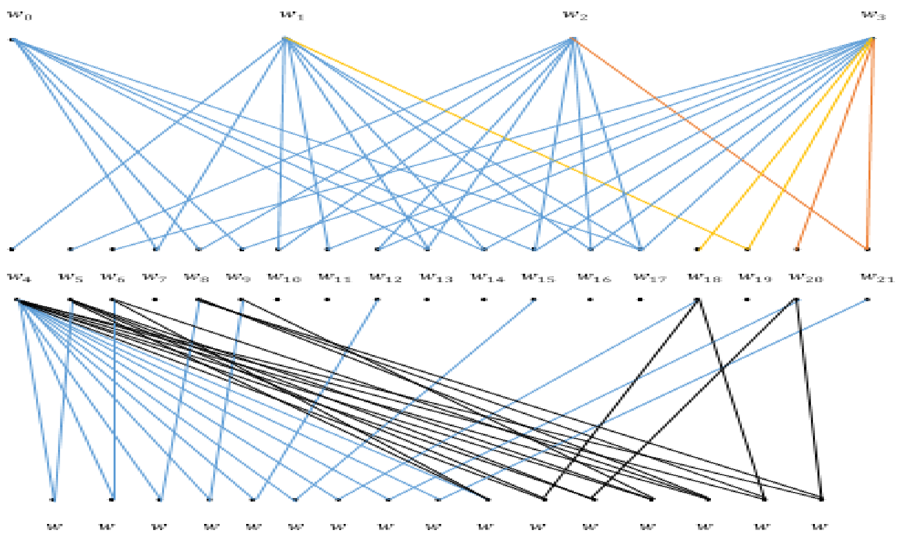

Figure (5) The generic Kripke model R_2

We have one element labelled by $\{p\}$ totally covered by $\{w_{18}, w_{19}\}$. In addition, one element labelled by $\{q\}$ totally covered by $\{w_{20}, w_{21}\}$.

So the total points generated by level-one are 262131.

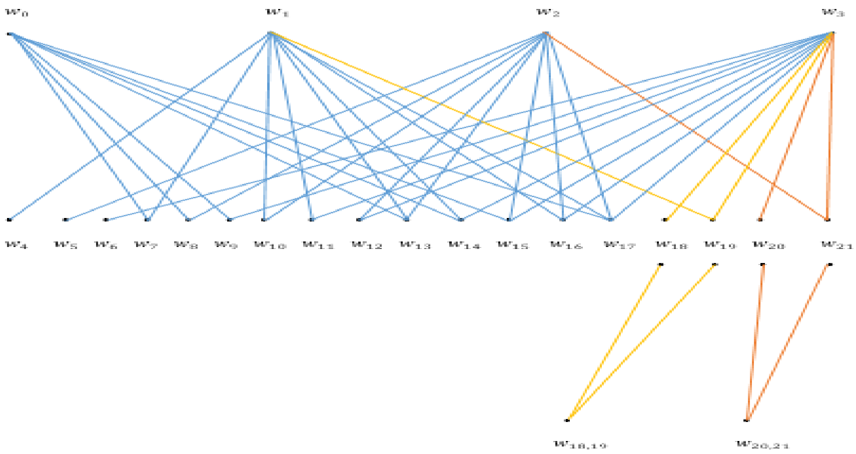

Figure (6) The generic Kripke model R_2

Now we describe the points generated by the elements from both level-zero and level one.

At first, there are three points generated by w_4 and w_0 or w_2 or w_4 respectively. And there are two points generated by w_4 and w_5 together with single points from level-zero, in this case w_0 or w_3 Similarly there are another six points generated by w_4 and either w_6 or w_{18} or w_{20} together with single points from level-zero. Also there are three points generated by w_4 ,w_5 , $w_6$together with $w_0$or w_4, w_5, w_{18} together with w_0 or w_4 , w_5 , w_{20} together with w_0. Finally there are four points generated by w_4 and two points

from level-one together with a single point from level-zero.

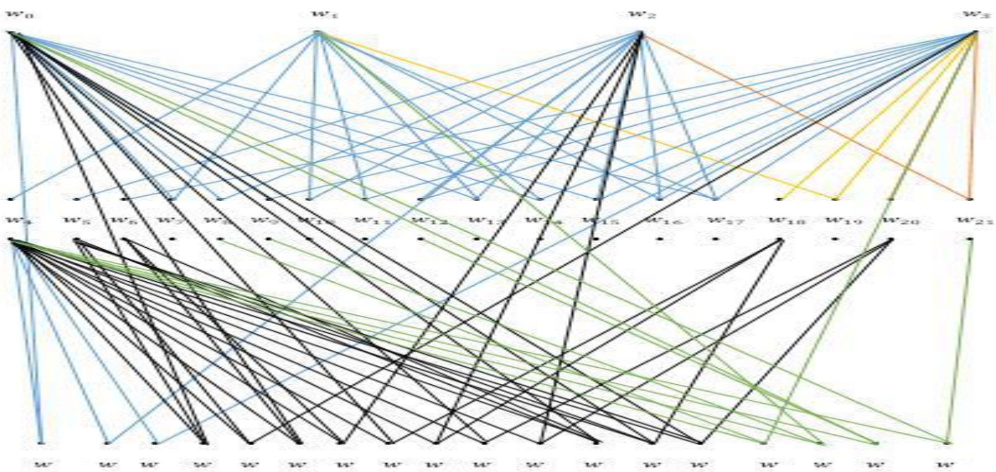

Figure (7) The generic Kripke model R_2

Uniformly, there are three points generated by w_5 *and* w_0 *or* w_1 *or* w_3 respectively. And there are two points generated by w_5 and w_6 together with single points from level-zero, in this case w_0 w_1 . Similarly, there are another four points generated by w_5 and either w_{18} *or* w_{20} together with single points from level-zero.

Finally, there are four points generated by w_5 and two points from level-one together with a single point from level-zero.

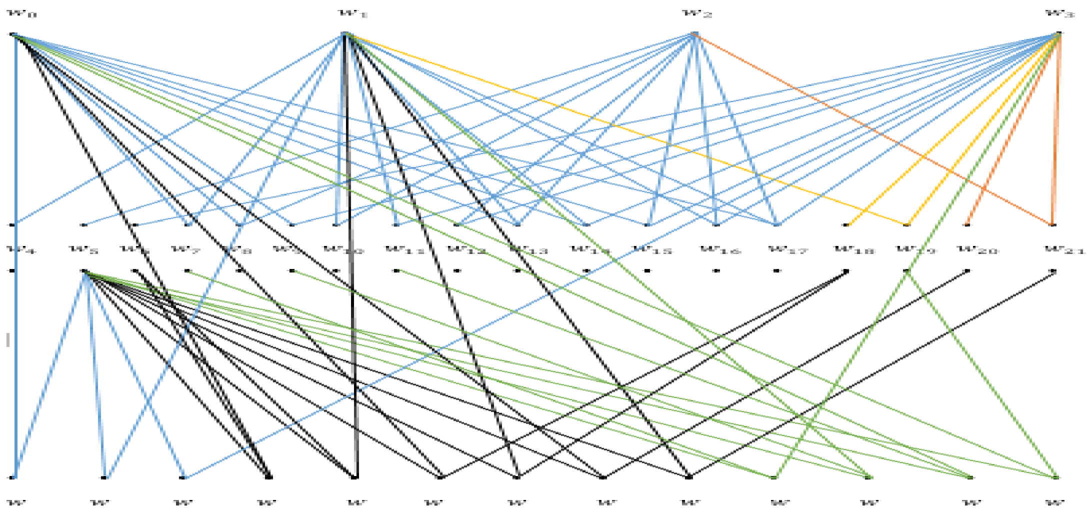

Figure (8) The generic Kripke model R_2

Similarly, there are three points generated by w_6 *and* w_0 *or* w_1 *or* w_2 respectively.

Finally, there are three points generated by w_6 and two points from level-one together with a single point from level-zero.

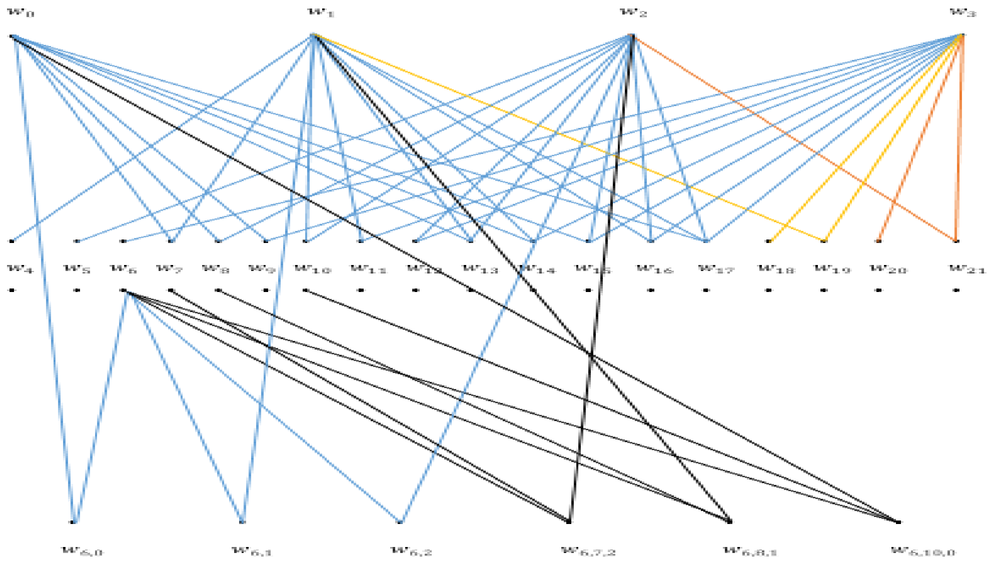

Figure (9) The generic Kripke model R_2

Now for the details for some special elements generated by w_{18},w_{20}

As for w_{18} there is an element which is covered by w_1 and w_{18} labelled by $\{P\}$.

Similarly, there is an element, that is covered by w_2 and w_{20} labelled by $\{q\}$.

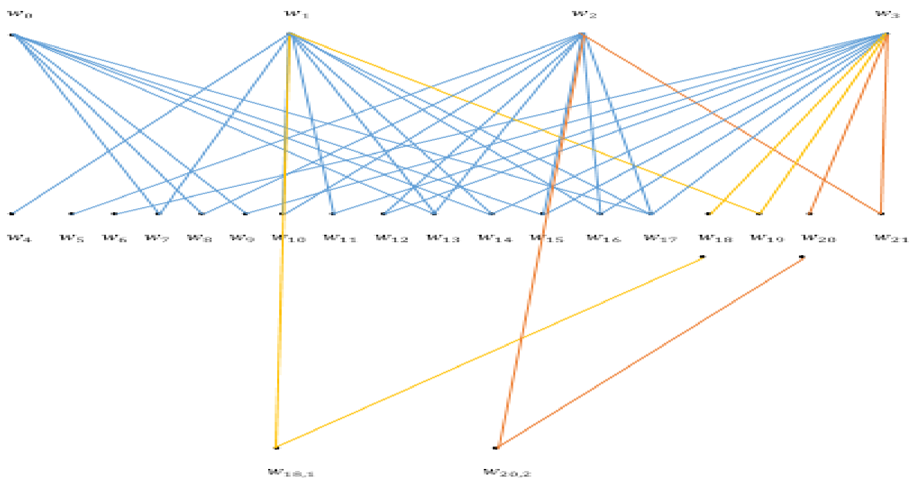

Figure (10) The generic Kripke model R_2

Some elements are generated from two parents, one of the parents is from level-zero and the other is a two-parent child from level one.

For example in figure (11) there are two points generated by w_7 and w_2 or w_3 respectively.

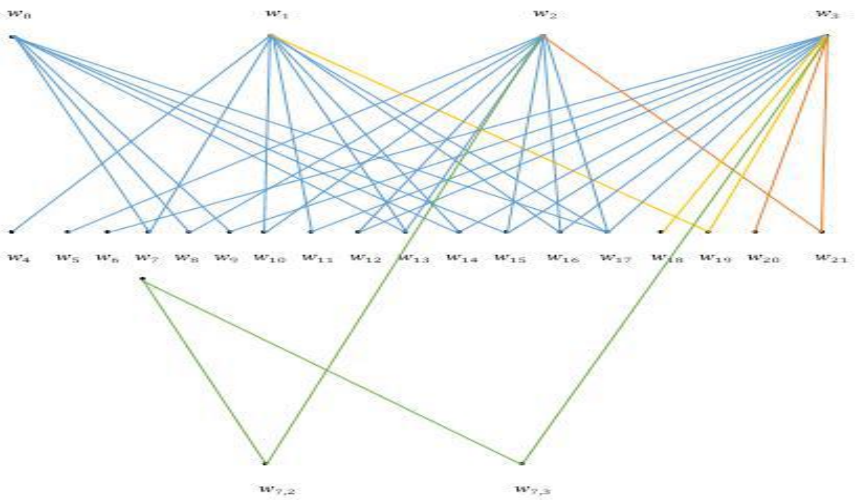

Figure (11) The generic Kripke model R_2

As a last example there are elements generated by a three-parent child from level one and one element from level zero, we can see that in figure (12) where w_3 and w_{13} generate a label in level two.

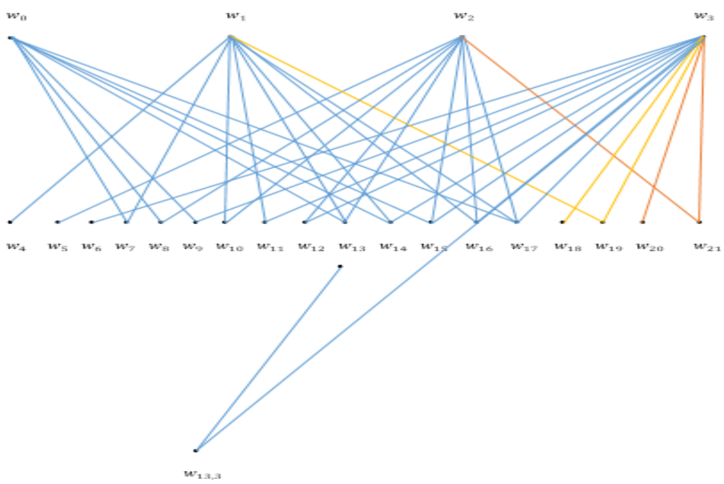

Figure (12) The generic Kripke model R_2.

7. Bellissima's construction for three generators

In the Kripke model R_3 on three generators p ,q and r there are eight possible labels and so eight elements w_0, w_1, w_2, w_3, w_4 , w_5, w_6, w_7 in level zero labelled by $\emptyset$,$\{p\}$, $\{q\}$, $\{r\}$, $\{p,q\}$, $\{p,r\}$, $\{q,r\}$, $\{p,q,r\}$ respectively.

In level one $2^8 - 2$ elements are labelled by $\emptyset$ and 48 labelled by non-empty subset of $\{p, q, r\}$.

In level one of R_3 there are fourteen elements labelled by $\{p\}$three of them are single-parent children from w_4 or w_5 *or* w_7 , six of them are two-parented children, three are

generated by w_1 and w_4 or w_5 or w_7,two are generated by w_4 and w_5 or w_7, one is generated by w_5 and w_7 , four of the elements are three-parented children, two are generated by w_1 and w_4 $with$ w_5 or w_7 , as well w_1 $with$ w_5 and w_7 or w_4 $with$ w_5 and w_7.
Finally the last points generated by w_1 , w_4, w_5 and w_7.

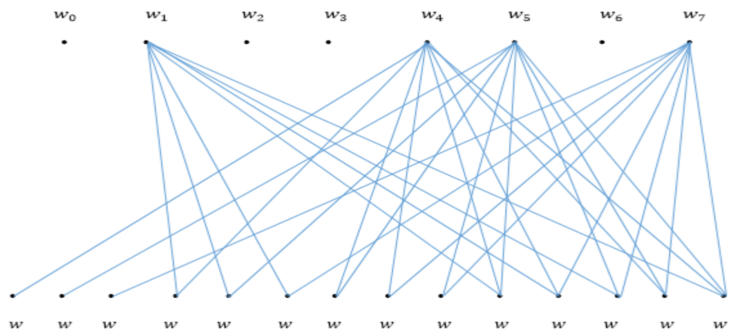

Figure (13) The generic Kripke model R_3

In this section we show the exact number of elements for level two of two generators for Bellissima's constructions and we did it in two ways.

8. Counting the elements manually

In (Elageili, & Truss, 2012) there was a remark that gave a formula that finds the number of elements in level one.

For all $\alpha < \omega$, $|lev_{\alpha,1}| = \sum_{j=0}^{\alpha} \frac{\alpha!}{j!(\alpha-j)!}\left[2^{2^{\alpha-j}}\right]$, where $\frac{\alpha!}{j!(\alpha-j)!}$ Is the number of subsets of $P_\alpha$of size j.

Therefor we knew that by using combinations we could find the number of elements in level two.

The elements in level two that are generated by the elements of level one only, are

$$\sum_{j=0}^{18} \frac{18!}{j!\,(18-j)!} - 12$$

=262131 elements

Where we exclude the fourteen labels of $\emptyset$ and we added the two labels of p and q that are covered by $\{w_{18}, w_{19}\}$ and $\{w_{20}, w_{21}\}$ respictavily.

Now for the elements that are generated by level one and zero. We started by finding the number of elements that are not a child of the parents in level zero.

For w_0 it is a parent for $w_7, w_8, w_9, w_{13}, w_{14}, w_{15}, w_{17}$ so it can not generate with these elements, therefor the number of elements that it can generate with is eleven elements.

And this gives us

$$\sum_{j=1}^{11} \frac{11!}{j!\,(11-j)!}$$

=2047 elements.

The element w_1 is a parent for $w_4, w_7, w_{10}, w_{11}, w_{13}, w_{14}, w_{16}, w_{17}, w_{19}$ that means that can generate with nine elements of level one, so this gives us

$$\sum_{j=1}^{9} \frac{9!}{j!\,(9-j)!}$$

=511 elements.

Similarly, w_2 is a parent for $w_5, w_8, w_{10}, w_{12}, w_{13}, w_{14}, w_{16}, w_{17}, w_{21}$ so it as well can generate with nine elements, so we get the same number of sets of combinations as we got from w_1 which is 511 elements.

For the element $w_3$it is a parent for $w_6, w_9, w_{11}, w_{12}, w_{14}, w_{15}, w_{16}, w_{17}, w_{18}, w_{19}, w_{20}, w_{21}$. The total of children is twelve, which means that it can generate with 6 elements and that gives us

$$\sum_{j=1}^{6} \frac{6!}{j!\,(6-j)!}$$

=63 elements.

Then the number of elements that are generated by two elements from level zero.

We start by the two elements w_0 and $w_1$they only can generate with $w_5, w_6, w_{12}, w_{18}, w_{20}, w_{21}$ from this the number of elements that they generate together with level one are

$$\sum_{j=1}^{6} \frac{6!}{j!\,(6-j)!}$$

=63 elements.

For the elements $w_0$and w_2 they can generate with $w_4, w_6, w_{11}, w_{18}, w_{19}, w_{20}$, so we get the same number of elements that we got from w_0 and w_1 and that is 63 elements.

Now for the elements $w_0$and $w_3$they only can generate with three elements w_4, w_5, w_{10} therefor the number of elements are given by

$$\sum_{j=1}^{3} \frac{3!}{j!\,(3-j)!}$$

=7 elements.

For the elements $w_1$and w_2 together they can generate with w_6, w_9, w_{18}, w_{20} and from these we get

$$\sum_{j=1}^{4} \frac{4!}{j!\,(4-j)!}$$

=15 elements.

For the two elements $w_1$and w_3 they can generate with w_5, w_8, so all combinations for them are

$$\sum_{j=1}^{2} \frac{2!}{j!\,(2-j)!}$$

=3 elements.

And we get the same number of elements when $w_2$and $w_3$generate together with level zero and that is due to that they only can generate with w_4, w_7 and that is 3 elements.
Finally, we find the number of elements that are generated by three elements from level zero and we start with w_0, w_1 and w_2 together they only can generate with w_6, w_{19}, w_{21}.

As for w_0, w_1 and $w_3$together they only can generate with one element and it is w_5.
At last for w_0, w_2 and w_3 they can only generate with $w_4$and these are given by

$$\sum_{j=1}^{3} \frac{3!}{j!\,(3-j)!} + 2\sum_{j=1}^{1} \frac{1!}{j!\,(1-j)!}$$

=9 elements.

The total number of elements in level two including the two elements shown in figure (10) are 265428 elements.

9.Counting the elements by programming using a Python code

The code used can be found in [13].

The results when we run the code:

i) For one generator :

Enter number of generators: 1

Enter elements separated by comma ie. p,q : p

Number of levels? : 5

LEVEL 0: 2 labels

LEVEL 1: 2 labels

LEVEL 2: 2 labels

LEVEL 3: 2 labels

LEVEL 4: 2 labels

LEVEL 5: 2 labels

Total labels: 12

['ϕ', {'p'}]

['ϕ', 'ϕ']

['ϕ', 'ϕ']

['ϕ', 'ϕ']

['ϕ', 'ϕ']

['ϕ', 'ϕ']

i) For two generators

Enter number of generators: 2

Enter elements separated by comma i.e. p, q : p, q.

Number of levels? : 2

LEVEL 0: 4 labels

LEVEL 1: 18 labels

LEVEL 2: 265428 labels

Total labels: 265450

['ϕ', {'p'}, {'q'}, {'q', 'p'}]

['ϕ', 'ϕ', 'ϕ', {'q'}, {'p'}, 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', {'p'}, 'ϕ', {'q'}, 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ']

['ϕ', {'q'}, 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', {'q'}, 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', {'p'}, 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', {'p'}, 'ϕ',...

Plus 225 pages of a word document file of labels.

Definition 9.1

A *pre-low ladder* L is an upwards closed subset of R_2, which is isomorphic to R_1.

Remark 9.1

The results that we get in level two shows that we have only two labels of ‘p’ and two labels of ‘q’ as you can see in lines six and seven of the execute.

These labels are in figures (9) and (12)

Therefore we get a pre-low ladder containing of only labels of ‘p’ and a pre-low ladder containing of only labels of ‘q’ which are isomorphic to R_1

ii) For three generators

Enter number of generators: 3

Enter elements separated by comma i.e. p, q : p, q, r

Number of levels? : 1

LEVEL 0: 8 labels

LEVEL 1: 302 labels

Total labels: 310

['ϕ', {'p'}, {'q'}, {'r'}, {'p', 'q'}, {'p', 'r'}, {'q', 'r'}, {'p', 'q', 'r'}]

['ϕ', 'ϕ', 'ϕ', 'ϕ', {'p'}, {'q'}, 'ϕ', {'p'}, {'r'}, 'ϕ', {'q'}, {'r'}, 'ϕ', {'p'}, {'q'}, {'r'}, {'p', 'q'}, {'p', 'r'}, {'q', 'r'}, 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', {'p'}, 'ϕ', {'p'}, 'ϕ', 'ϕ', {'p'}, 'ϕ', 'ϕ', {'q'}, 'ϕ', 'ϕ', {'q'}, 'ϕ', {'q'}, 'ϕ', 'ϕ', {'r'}, 'ϕ', {'r'}, 'ϕ', {'r'}, 'ϕ', {'p'}, 'ϕ', {'q'}, 'ϕ', {'p'}, {'q'}, {'p', 'q'}, 'ϕ', {'r'}, 'ϕ', {'p'}, {'r'}, {'p', 'r'}, 'ϕ', {'q'}, {'r'}, {'q', 'r'}, 'ϕ', {'p'}, 'ϕ', 'ϕ', {'p'}, 'ϕ', 'ϕ', {'p'}, 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', {'q'}, 'ϕ', {'q'}, 'ϕ', 'ϕ', 'ϕ', {'q'}, 'ϕ', 'ϕ', 'ϕ', 'ϕ', {'r'}, 'ϕ', {'r'}, 'ϕ', {'r'}, 'ϕ', 'ϕ', {'p'}, 'ϕ', {'q'}, 'ϕ', {'r'}, 'ϕ', {'p'}, 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', {'q'}, 'ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', {'r'}, 'ϕ', 'ϕ', ..., 'ϕ'].

iii) For four generators

Enter number of generators: 4

Enter elements separated by comma i.e. p, q : p, q, r, s

Number of levels? : 1

LEVEL 0: 16 labels

LEVEL 1: 66642 labels

Total labels: 66658

['ϕ', {'p'}, {'q'}, {'r'}, {'s'}, {'q', 'p'}, {'r', 'p'}, {'s', 'p'}, {'q', 'r'}, {'q', 's'}, {'r', 's'}, {'q', 'r', 'p'}, {'q', 's', 'p'}, {'r', 's', 'p'}, {'q', 'r', 's'}, {'q', 'r', 's', 'p'}]

['ϕ', 'ϕ', 'ϕ', 'ϕ', 'ϕ', {'q'}, {'p'}, 'ϕ', {'r'}, {'p'}, 'ϕ', {'s'}, {'p'}, 'ϕ', {'q'}, {'r'}, 'ϕ', {'q'}, {'s'}, 'ϕ', {'r'}, {'s'}, 'ϕ', {'q'}, {'r'}, {'p'}, {'q', 'r'}, {'q', 'p'}, {'r', 'p'}, 'ϕ', {'q'}, {'s'}, {'p'}, {'q', 's'}, {'q', 'p'}, {'s', 'p'}, 'ϕ', {'r'}, {'s'}, {'p'}, {'r', 's'}, {'r', 'p'}, {'s', 'p'}, 'ϕ', {'q'}, {'r'}, {'s'}, {'q', 'r'}, {'q', 's'}, {'r', 's'}, 'ϕ', {'q'}, {'r'}, {'s'}, {'p'}, {'q', 'r'}, {'q', 's'}, {'q', 'p'}, {'r', 's'}, {'r', 'p'}, {'s', 'p'}, {'q', 'r', 's'}, {'q', 'r', 'p'}, {'q', 's', 'p'}, {'r', 's', 'p'}, 'ϕ', {'p'}, 'ϕ', {'p'}, 'ϕ', {'p'}, 'ϕ', 'ϕ', 'ϕ', 'ϕ', {'p'}, 'ϕ', {'p'}, 'ϕ', {'p'}, 'ϕ', 'ϕ', {'p'}, 'ϕ', 'ϕ', 'ϕ', {'q'}, 'ϕ', 'ϕ', 'ϕ', {'q'}, 'ϕ', {'q'}, 'ϕ', 'ϕ', {'q'}, 'ϕ', {'q'}, 'ϕ', 'ϕ', {'q'}, 'ϕ', {'q'}, 'ϕ', 'ϕ', 'ϕ', {'r'}, 'ϕ', 'ϕ', {'r'}, 'ϕ', 'ϕ', {'r'}, 'ϕ', {'r'}, 'ϕ', 'ϕ', {'r'}, 'ϕ', {'r'}, 'ϕ', {'r'}, 'ϕ', 'ϕ', 'ϕ', {'s'}, 'ϕ', 'ϕ', {'s'}, 'ϕ', {'s'}, 'ϕ', 'ϕ', {'s'}, 'ϕ', {'s'}, 'ϕ', {'s'}, 'ϕ', {'s'}, 'ϕ', {'p'}, 'ϕ', {'p'}, 'ϕ', {'q'}, 'ϕ', {'q'}, 'ϕ', 'ϕ', {'q'}, {'p'}, {'q', 'p'}, 'ϕ', {'q'}, {'p'}, {'q', 'p'}, 'ϕ', {'p'}, 'ϕ', {'q'}, 'ϕ', {'q'}, {'p'}, {'q', 'p'}, 'ϕ', {'p'}, 'ϕ', {'r'}, 'ϕ', 'ϕ', {'r'}, 'ϕ', {'r'}, {'p'}, {'r', 'p'}, 'ϕ', {'p'}, 'ϕ', {'r'}, {'p'}, {'r', 'p'}, 'ϕ', {'r'}, 'ϕ', {'r'}, {'p'}, {'r', 'p'}, 'ϕ', 'ϕ', {'s'}, 'ϕ', {'s'}, 'ϕ', {'p'}, 'ϕ', {'s'}, {'p'}, {'s', 'p'}, 'ϕ', {'s'}, {'p'}, {'s', 'p'}, 'ϕ', {'s'}, 'ϕ', {'s'}, {'p'}, {'s', 'p'}, 'ϕ', {'q'}, 'ϕ', {'r'}, 'ϕ', {'q'}, {'r'}, {'q', 'r'}, 'ϕ', {'q'}, 'ϕ', {'r'}, 'ϕ', {'q'}, {'r'}, {'q', 'r'}, 'ϕ', {'q'}, {'r'}, {'q', 'r'}, 'ϕ', {'s'}, 'ϕ', {'q'}, 'ϕ', {'q'}, {'s'}, {'q', 's'}, 'ϕ', {'s'}, 'ϕ', {'q'}, {'s'}, {'q', 's'}, 'ϕ', {'q'}, {'s'}, {'q', 's'}, 'ϕ', {'r'}, 'ϕ', {'s'}, 'ϕ', {'r'}, {'s'}, {'r', 's'}, 'ϕ', {'r'}, {'s'}, {'r', 's'}, 'ϕ', {'r'}, {'s'}, {'r', 's'}, 'ϕ', {'q'}, {'p'}, {'q', 'p'}, 'ϕ', {'r'}, {'p'}, {'r', 'p'}, 'ϕ', {'q'}, {'r'}, {'q', 'r'}, 'ϕ', {'q'}, {'r'}, {'p'}, {'q', 'r'}, {'q', 'p'}, {'r', 'p'}, {'q', 'r', 'p'}, 'ϕ', {'s'}, {'p'}, {'s', 'p'}, 'ϕ', {'q'}, {'s'}, {'q', 's'}, 'ϕ', {'q'}, {'s'}, {'p'}, {'q', 's'}, {'q', 'p'}, {'s', 'p'}, {'q', 's', 'p'}, 'ϕ', {'r'}, {'s'}, {'r', 's'}, 'ϕ', {'r'}, {'s'}, {'p'}, {'r', 's'}, {'r', 'p'}, {'s', 'p'}, {'r', 's', 'p'}, 'ϕ', {'q'}, {'r'}, {'s'}, {'q', 'r'}, {'q', 's'}, {'r', 's'}, {'q', 'r', 's'}, 'ϕ', 'ϕ',.....

Plus 56 pages of a word document file of labels.

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