The Leading Asymptotic Term for the Gamma Function of Matrices with a Large Parameter

Hussein Shat
General Department, Faculty of Management Bright Star University, Libya
hussein.saliem@bsu.edu.ly

Salah Hamad
Faculty of Sciences,
University of Benghazi, Libya
salah.hamad@uob.edu.ly

Abstract:

We derive the leading asymptotic term for the Gamma function of matrices. The leading asymptotic approximation to derivatives of any order of the scalar Gamma function is also obtained.

Keywords: Special functions; Asymptotic Approximations; Gamma function; Gamma function of matrices.

الملخص

نشتق الحد المقارب الرئيسي لدالة غاما للمصفوفات . كما تم الحصول على التقريب االمقارب الرئيسي لمشتقات أي رتبة لدالة غاما القياسية

الكلمات المفتاحية: الدوال الخاصة ، تقريبات مقاربة ، دالة غاما ، دالة غاما للمصفوفات

1. Introduction:

The purpose of this paper is to derive the following asymptotic results for the Gamma matrix function $\Gamma(\theta Q)$ with θ is a large parameter. Scalar functions and matrix functions with a large argument are common place in theory and applications. Details follow.

Proposition 1.1 Suppose

(i) the eigenvalues λ_k , k = 1, 2, . . . , r of $Q \in c^{r\times r}$ satisfy $\mathrm{Re}\lambda_k > 0$

(ii) $\theta \in (0, \infty)$

Then, for Q fixed and as $\theta \to \infty$ we have for a certain constant invertible

T that $\Gamma(\theta$ Q) ~ T(w_{xy})T^{-1}. This asymptotic relation ~ between matrices

holds in the following sense. The matrix (w_{xy}) is such that, when $> y$,

we have w_{xy} = 0 , and when $x \leq y$

$$w_{xy}= \frac{\theta^j \ d^j}{j_1 \cdot d(\theta\lambda_k)^j} \ \Gamma(\theta\lambda_k) \qquad (1.1)$$

$$\sim \sqrt{\frac{2\Pi}{\theta\lambda_k}} \ e^{\theta\lambda_k \log(\theta\lambda_k) - \theta\lambda_k} \ Ln^j \ \theta \ \text{, where } 0 \leq j \leq m_k - 1$$

$\Gamma(z)$ is the celebrated Euler Gamma function and $\Gamma(Q)$ is the Matrix Gamma

function of the r × r matrix Q .

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} \, dt \, , \ t^{p-1} = \exp(\,(P-1)\ln t) \qquad (1.2)$$

Remark We adopt in equation (1.1) the convention of using the principal value of the complex valued logarithmic function.

$$log(z) \coloneqq ln|z| + i\alpha \, , \ -\pi < \alpha < \pi \, , \ \sqrt{\frac{2\pi}{\theta\lambda_k}} := exp\left(\frac{1}{2} Log\left(\frac{2\pi}{\theta\lambda_k}\right)\right) \qquad (1.3)$$

Central to proving this result is the following lemma on the leading asymptotic term of all higher order derivatives of the scalar Gamma function.

Lemma 1.2 Let λ ∈ D = { $|\lambda| \ e^{i\alpha}: 0 < |\lambda_0| < |\lambda| \, , \ -\pi + \delta \leq \alpha \leq \pi - \delta$ } ,

where λ_0 and $0 < \delta < \pi$ are certain constants. Then we have

$$\Gamma^{(\nu)}(\lambda) \sim e^{\left(\lambda - \frac{1}{2}\right) Log\, \lambda - \lambda + \frac{1}{2}\ln 2\pi} \ [Log\ \lambda]^\nu \ \text{ as } \lambda \to \infty \, , \ \nu \in \mathbb{N} \, . \qquad (1.4)$$

Given the myriad of applications of the scalar Gamma function and its derivatives in mathematics and mathematical physics the above lemma 1.2 could be useful not only for the sake of proving Proposition 1.1, but also in numerous other applications. We could not find this lemma explicitly in the voluminous literature. It does not appear explicitly in any of the textbooks that are listed in our references. Compare e.g. with Rainville E . F. [17],

Andrews G. et al [1], Olver F. W. J. [16] and Wong R. [24]. Proposition 1.1 is motivated by the generalizations and extensions of scalar special functions to matrix special functions that took place in the past two decades. The Gamma matrix function, whose eigenvalues are all in the right open half - plane is introduced and studied in Jódar L , Cortés J [12] for matrices in $\mathbb{C}^{r\times r}$.

Hermite matrix polynomials are introduced by Jódar L . et al [11] and some of their properties are given in Defez E , Jódar L. [5]. Other classical orthogonal polynomials as Laguerre and Chebyshev have been extended to orthogonal matrix polynomials, and some results have been investigated in Jódar L, Sastre J. [14] and Defez E, Jódar L. [6]. Relations between the Beta, Gamma and the Hypergeometric matrix function are given in J´odar L, Cortés J. G. [12]. , Salah [20] and Batahan R. S. [3]. These special functions of matrices have become an important tool in both theory and applications see eg Salah [19,21,22]. A few examples follow. In (1998). Jódar and Cortes [13] studied the Hypergeometric matrix series

$$F(A,B;C_{;}z) = I + \sum_{n=1}^{\infty}(A)_n(B)_n\,(C)_n^{-1}\,\frac{z^n}{n!} \qquad (1.5)$$

with , B and C , r ×r matrices with $B\ C = C\ B$. They proved among other things that:

(a) $F(A,B;C_{;}z)$ satisfies the hypergeometric matrix equation

$$z(1-z)\,w^{(2)} - z\,Aw^{(1)} + w^{(1)}\,(C - z\,(B+I)) - A\,w\,B = 0,\quad 0 \le |\,z\,| < 1 \qquad (1.6)$$

(b) with $B = -\,n\,I$ where n is a natural number, the equation (1.6) reduces to

$$z(1-z)\,w^{(2)} - z\,Aw^{(1)} + w^{(1)}\,(C - z\,(n-1)\,) - n\,A\,w = 0 \qquad (1.7)$$

that possesses a matrix polynomial solutions of degree n. The matrix Gamma function occurs in the integral with matrix argument

$$F(A,B;C_{;}z)$$

$$= \left(\int_0^1 (1-tz)^{-1}\ t^{\beta-1}\,(1-t)^{C-B-I}dt\right)\quad \Gamma^{-1}(\beta)\Gamma^{-1}(\ -B\ )\,\Gamma(c) \qquad (1.8)$$

With A = O we obtain the special case of

$$F(0,B;C_{;}z) = \left(\int_0^1\ t^{\beta-1}\,(1-t)^{C-B-I}dt\right)\quad \Gamma^{-1}(\beta)\Gamma^{-1}(\ -B\ )\,\Gamma(c) \qquad (1.9)$$

that is the beta function with matrix arguments. It is evident from equation (1.8) that a study of the limiting behavior of $F(A,B;C_{;}z)$ as C = θQ becomes large, requires the services of proposition 1.1. Matrices of the form with θ a large parameter occur frequently in theory and applications and could require the asymptotic approximation of integrals with a matrix argument.

The right hand of equation (1.8) reveals that the asymptoic approximation of the integrals representing the Gamma function, play a special role in the theory of special functions of matrices. Proposition 1.1 and Lemma 1.2 are also relevant to the study of a singular perturbation problem of the equation (1.7) For example, consider C a large matrix $C = \theta$Q such that $\theta \to \infty$. It is readily recognized that if we set $\in := \theta^{-1}$ then the differential equation (1.7) becomes

$$\epsilon\big(z(1 - z)\, w^{(2)} - z\, Aw^{(1)}\big) + w^{(1)}\, \mathrm{Q} + \in \big(\, w^{(1)}(\, z\,(n-1)\ I\,\big) + nAw = 0 \qquad (1.10)$$

Formally, for $\in$=0 we obtain a "reduced" hypergeometric equation $w_0^{(1)}\mathrm{Q} = \mathrm{O}$. See e.g. Wasow [23] and Nayfeh [15] for singular perturbations problems. The order of presentation in this article is as follows. In section 2 we provide a proof to Lemma 1.2 and proposition 1.1, and in section 3 we provide an example.

2 The Main Result

2.1 Proof of Lemma 1.2

It is well known, see e.g. Arfken George B, Weber Hans J. [2] and Gradshteyn $I.S$. et at [8], that

$$\Gamma(\lambda) = e^{\left(\lambda-\frac{1}{2}\right)Log\,\lambda - \lambda + \frac{1}{2}\ln 2\pi}\ [1+s\,] \ \text{ or } \ \Gamma(\lambda) = e^{g(\lambda)}\ [1+s\,] \qquad (2.1)$$

Where

$$g\,(\lambda) = (\lambda - \tfrac{1}{2})log\,\lambda - \lambda + \tfrac{1}{2}\ Ln2\pi \ \text{ and } \ \mathrm{S} = \frac{1}{12\lambda} + \frac{1}{288\lambda^2} + \Delta \qquad (2.2)$$

and where Δ is an analytic function in the sector D such that

$$\Delta \sim \sum_{\kappa=3}^{\infty} a_k \lambda^{-k} as\,\lambda \to \infty, \lambda \in D$$

A result of Ritt [18] states the following. Let $f(\lambda)$ be holomorphic in a sector D defined by the inequalities $0 < \lambda_0 \le |\lambda|$, $\alpha_1 \le arg\,\lambda \le \alpha_2$ with $\alpha_2 > \alpha$ being real numbers. $let\ f(\lambda) \sim \sum_{r=0}^{\infty} a_r \lambda^{-r}$ as $\lambda \to \infty$, $\lambda \in$ D, then $f^{(1)} \sim \sum_{r=0}^{\infty} r a_r \lambda^{-r-1}$ as $\lambda \to \infty$, in every proper sub sector D^* : $\alpha_1 < \alpha_1^* \le \arg\lambda \le \alpha_2^* < \alpha_2$ where α_1^* and α_2^* are certain real numbers. Consequently, we have

$$\Delta^{(\nu)} \sim \Sigma(\, a_k \lambda^{-k})^{(\nu)} \text{ as } \lambda \to \infty\ , \lambda \in \mathrm{D} \qquad (2.3)$$

By Leibniz formula

$$\Gamma^{(\nu)}(\lambda) = \sum_{L=0}^{\nu} \binom{\nu}{L} \left(e^{g(\lambda)}\right)^{(L)} [1+s]^{\nu-L} \qquad (2.4)$$

Considering the L derivatives of $e^{g(\lambda)}$ with respect to λ we have ,

$$\left(e^{g(\lambda)}\right)^{(L)} = e^{g(\lambda)} [\, g^{(1)}(\lambda)]^{L} + L e^{g(\lambda) g^{(1)}}(\lambda) g^{(L-1)}(\lambda) + \ldots + e^{g(\lambda)}\, g^{(L)}(\lambda) \qquad (2.5)$$

Now let t be the transposition operation. In the sequel we denote by $l^t = (l_1, l_2, \cdot\cdot\cdot, l_n)^t$,

$\nu^t = (1, 2, \cdot\cdot\cdot, \nu_n)^t$ where $n \in \mathbb{N}$, the transposition of certain column vectors in $\mathbb{R}^{n\times n}$and we denote $\alpha(l, \nu)$ certain coefficients to be elaborated upon in the sequel. We also put L for the inner product of l and ν such that

$$L = \langle\, l, \nu \,\rangle = l_1 + l_2 2 + \cdot\cdot\cdot + l_n \nu_n \quad with \quad l_i < L \quad \text{for all } 1 \le i \le n. \qquad (2.6)$$

A special case of the $Fa\acute{a}$ di Brun$\acute{o}$s formula, see e.g. Johnson W. P. and Bell E. T. [10] and [4], we have

$$\left(e^{g(\lambda)}\right)^{(L)} = e^{g(\lambda)} [\, g^{(1)}(\lambda)]^{L} + \sum_{finitesum} \alpha(l,\nu)\, e^{g(\lambda)} [\, g^{(1)}(\lambda)]^{l_1} \ldots [\, g^{(\nu_n)}(\lambda)]^{l_n}$$

$$= e^{g(\lambda)} [\, g^{(1)}(\lambda)]^{L} \left[\, 1 + \sum_{finitesum} \alpha(l,\nu) \frac{[\, g^{(1)}(\lambda)]^{l_1}}{[\, g^{(1)}(\lambda)]^{L}} \ldots [\, g^{(\nu_n)}(\lambda)]^{l_n} \right] \qquad (2.7)$$

where the finite sum is taken subject to (2.6). Similarly, we calculate the ν_nth derivatives of $g(\lambda)$ with respect to λ, we have, when $\nu_n = 1$. $g^{(1)}(\lambda) = L\text{og}\,\lambda - \frac{1}{2}\lambda^{-1}$, and when $\nu_n \ge 2$

$g^{(\nu_n)}(\lambda) = a_{\nu_n}\ \lambda^{-(\nu_n - 1)} + b_{\nu_n}\ \lambda^{-\nu_n}$, where

$a_{\nu_n} = (-1)^{\nu_n}(\nu_n - 2)!$, $b_{\nu_n} = \frac{(-1)^{\nu_n}(\nu_n - 1)!}{2}$ are certain constants. Since

$[\, g^{(1)}(\lambda)]^{-L+l_i} = (\, L\text{og}\,\lambda - \frac{1}{2}\lambda^{-1})^{-L+l_i} = (\, L\text{og}\,\lambda)^{-L+l_i}[\, 1 + O(\,\lambda^{-1}(\, L\text{og}\,\lambda)^{-1})\,]$.

Therefore equation (2.7) becomes $e^{g(\lambda)} [\, g^{(1)}(\lambda)]^{L} [1 + s_1]$ where $s_1 = O\{\lambda^{-p^2}\}$ for some $p \in \mathbb{R}$. Now we can write equation (2.4) as

$$\sum_{L=0}^{\nu} \binom{\nu}{L}\ e^{g(\lambda)} \left[\psi^{(1)}(\lambda)\right]^{L} [1 + s_1]\ [1+s]^{(\nu - L)}$$

$$= \left(e^{g(\lambda)}\right)^{(\nu)} \left[\, 1 + \frac{1}{\left(e^{g(\lambda)}\right)^{(\nu)}} \sum_{L=0}^{\nu-1} \binom{\nu}{L} e^{g(\lambda)} [\, g^{(1)}(\lambda)]^{L} [1+s_1] [1+s]^{(\nu-L)} \right] \qquad (2.8)$$

$= \left(e^{g(\lambda)}\right)^{(\nu)}[\ 1 + O(\ \lambda^{-p^2})\ ] = e^{\left(\lambda-\frac{1}{2}\right)Log\,\lambda\,-\,\lambda+\frac{1}{2}\ln 2\pi}\,[\ Log\,\lambda - \frac{1}{2}\,\lambda^{-1}]^{\ \nu}\ [\ 1 + O(\ \lambda^{-p^2})\ ]$

and the conclusion (1.4) follows □.

Remark $f(\lambda) \sim h(\lambda)$ does not automatically imply $f^{(\nu)} \sim h^{(\nu)}$ as $\lambda \to \infty$ for $\nu \in \mathbb{N}$, unless we have special conditions as given in lemma 1.2. For example

$$f(\lambda) = [\ \lambda + \ \lambda^{-1}\, sin(\lambda^2)\ ] \sim \lambda = g(\lambda)$$

However, $f^{(1)}(\lambda) = 1 - \lambda^{-2}\ sin(\lambda^2\ ) + 2cos(\lambda^2\ )$ is not asymptotic anymore to

$1 = g^{(1)}(\lambda)$ as $\lambda \to \infty$

2.2 Proof of Proposition 1.1

By the Jordan Canonical form, it is well known, see e.g. Higham [9], that for

any matrix $Q \in \mathbb{C}^{rxr}$ there exists a constant invertible matrix T such that

$$Q = TJT^{-1} = T\,diag(J_1\,,\ j_2\,, \cdots ,\ J_8)\,T^{-1}\ ,\ S \in \mathbb{N} \qquad (\ 2.9\ )$$

Where

$$J_k(\lambda_k) = \lambda_k I_{m_k} + H_{m_k}\ , \quad 1 \le k \le S \qquad (\ 2.10\ )$$

$m_1 + m_2 + \cdots + m_8 = r$, I_{m_k} is an identity matrix of size $m_k \times m_k$ and

$$H_{m_k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix},\ of\ size\ m_k\ x\ m_k. \qquad (\ 2.11\ )$$

Consider the matrix θQ. By Jódar L. and Corté J. G., [12], $\Gamma(\theta Q)$ is well defined, therefore by e.g. Higham [9] we have

$$\Gamma(\theta Q) = T\Gamma(\theta Q)T^{-1} = Tdiag(\theta J_1, \theta J_2, \cdots, \theta j_8)T^{-1} \qquad (\ 2.12\ )$$

Consider

$$\Gamma(\theta J_k) = \int_0^\infty e^{-t}\ \ t^{\theta j_k - I}\ dt \qquad (\ 2.13\ )$$

Observe that

$$t^{\theta j_k - I} = t^{(\theta\lambda_k - 1)} \left[I_{m_k} + \sum_{j=1}^{m_k - 1} \frac{(\theta H_{m_k} \ln t)^j}{J!} \right],$$

Thus the integral (2.13) becomes

$$\Gamma(\theta J_k) = \int_0^{\infty} e^{-t}\ t^{(\theta\lambda_k - 1)} \sum_{j=0}^{m_k - 1} \frac{(\theta H_{m_k} \ln t)^j}{J!}\, dt\ . \qquad (2.14)$$

The typical term of the integral (2.14) is

$$\int_0^{\infty} e^{-t}\ t^{\theta\lambda_k - 1} \frac{(\theta\ \ln t)^j}{J!}\, \mathrm{d}t = \frac{\theta^j d^j}{J! d(\theta\lambda_k)^j} \Gamma(\theta\lambda_k)\ .$$

By lemma 1.2 we have,

$$\frac{\theta^j d^j}{J! d(\theta\lambda_k)^j} \Gamma(\theta\lambda_k) \sim e^{\left(\theta\lambda_k - \frac{1}{2}\right)\mathrm{Log}\,(\theta\lambda_k) - \theta\lambda_k + ln 2\pi} \ln^j \theta$$

Thus

$$\frac{\theta^j d^j}{J! d(\theta\lambda_k)^j} \Gamma(\theta\lambda_k) \sim \sqrt{\frac{2\pi}{\theta\lambda_k}}\quad e^{\theta\lambda_k\,\mathrm{Log}\,(\theta\lambda_k) - \theta\lambda_k} \ln^j \theta\ ,$$

3 An example

For an application of proposition 1.1 consider the matrix $Q \in \mathbb{C}^{4\times4}$ together with it is Jordan canonical form J

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -6 & 4 \end{bmatrix},\ J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ \text{obtained with } T, T^{-1} \text{such that},$$

$$T = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix},\ T^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$\Gamma(\theta Q) = T\ \Gamma(\ \theta J\ )\ T^{-1}$

$$= T\left(\ \Gamma(\theta)\ I_4 + \frac{\theta d\Gamma(\theta)}{d\theta} H_4 + \frac{\theta^2 d^2\Gamma(\theta)}{2 d\theta^2} {H^2}_4 + \frac{\theta^3 d^3\Gamma(\theta)}{6 d\theta^3} {H^3}_4\right) T^{-1}$$

By proposition 1.1 we get as $\theta \to \infty$

$$\Gamma\left(\theta\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 4 & -6 & 4\end{bmatrix}\right) \sim \sqrt{\frac{2\pi}{\theta}}$$

$$e^{\theta\,\mathrm{Ln}\,(\theta)-\,\theta}\begin{bmatrix}1 & -1 & 1 & -1\\ 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 2 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & Ln\theta & \frac{Ln^2\theta}{2!} & \frac{Ln^3\theta}{3!}\\ 0 & 1 & Ln\theta & \frac{Ln^2\theta}{2!}\\ 0 & 0 & 1 & Ln\theta\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 1 & -2 & 1\\ -1 & 3 & -3 & 1\end{bmatrix}$$

Conclusion

In summary, derive the asymptotic results for the Gamma matrix function with a large parameter is investigated in this study. The results show that Proposition 1.1 and Lemma 1.2 are also relevant to the study of a reduced hypergeometric equation for singular perturbations problems. The results demonstrated in this work provide a perspective on the importance of special functions of matrices in mathematics and mathematical physics.

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