On Review of Optimal Strong Stability Preserving Runge-Kutta Method (SSPRK) of Order Four

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Abstract

The explicit strong stability preserving Runge-Kutta method of four-stages fourth-order, SSPRK (4,4), is introduced in this work. The efficiency of SSPRK (4,4) is compared with that of the well-known optimal strong stability preserving Runge-Kutta of four-stages fifth-order SSPRK (5,4) introduced by Ruuth (2006) without mathematical prove to SSPRK(4,4). Results show that four-stages fourth-order Runge-Kutta, RK(4,4) is very efficient compared to five-stages fourth-order Runge-Kutta, RK(5,4) method. However, RK(4,4) has a lower effective SSP than RK(5,4).

Keywords: Finite element, strong stability preserving Runge-Kutta (SSPRK).

الملخص

هذا البحث يقدم طريقة رنج- كوتا الصريحة رباعية المراحل من الدرجة الرابعة ذات حفظ قوي للاستقرارية SSPRK (4,4) و مقارنة كفاءتها مع كفاءة SSPRK (5,4) المقترحة من راوث (2006) بدون اثبات رياضي لـ SSPRK(4,4). اوضحت النتائج ان RK(4,4) كفؤة جداً مقارنة بـ RK(5,4) ومع ذلك فإن هذه الطريقة لها effective SSP اصغر من ذلك الخاص بـ RK(5,4).

كلمات مفتاحية: طريقة العناصر المنتهية، معادلات تفاضلية، طريقة رانج كوتا قوية الاستقرارية SSPRK .

1. Introduction

‘Development of strong stability preserving SSP methods was historically motivated in two ways, and developed by two groups: one focusing on hyperbolic partial differential equations, the other focusing on ordinary differential equations’ (Gottlieb, Ketchson & Shu 2011). In this work, we restricted ourselves to the first group. SSP time discretization methods were first developed by Shu (1988) and by Shu and Osher (1988) were called total variation diminishing TVD time discretization. The explicit strong stability preserving (SSP) Runge–Kutta methods have been employed with a successful wide range of high-order numerical spatial discretization methods.

These methods are used for hyperbolic conservative laws including among others, the essentially non-oscillatory (ENO) finite volume and finite difference schemes (Shu and Osher, 1989), the weighted essentially non-oscillatory (WENO), finite volume and finite difference schemes (Jiang & Shu, 1996), spectral difference (SD) methods (Zhoa & Wang, 2010), spectral finite volume (SV) method (Sun & Wang, 2004), and the Runge–Kutta discontinuous Galerkin (RKDG) method (Cockburn & Shu,1989).

Time discretization of hyperbolic partial differential equation, which is ordinary differential equation (ODE), is solved mostly by using finite difference or finite element methods with the final form is Butcher or Shu-Osher form.

In that works, the main objective is the consistency requirement that is $\sum_{k=0}^{k=i-1} \alpha_{ik} = 1$, i = 1, 2 ,..s, where α_{ik} is the coefficients of state variables of previous steps with non-negative coefficients and s is the stage or step. The explicit strong stability preserving Runge-Kutta methods of fourth order-four steps SSPRK (4,4) is introduced and compared with a will know optimal Runge-Kutta of order four and five steps explicit Runge-Kutta (5,4) Ruuth method (Ruuth, 2006).

This work is organized as follows. Section 2 presented the time discretization by using finite element methods. Section 3 is the numerical solution, is used to compared the L_1 norm of the new method with optimal SSPRK (5,4) and SSPRK(4,3).

2. The mathematical model and time discretization method (finite element method)

Consider the governing equations written in conservation form,

$$\frac{\partial u}{\partial t} + \nabla . F = 0, \text{ in } \Omega \times (0,T). \qquad (2.1)$$

Equation 2.1 equipped with initial and boundary conditions, where u is the state variable (conservative variable) and F is the flux. The spatial part can be treated by using any aforementioned high order methods of solving spatial part of the governing equations. For example in DG method, the equation defining the spatial approximation solution after inverting the mass matrix can be written in ODE form as

$$\frac{du}{dt} = L(u(x)) = R(u(x)), \qquad (2.2)$$

where L(u(x)) is the spatial discretization residual. Then the resulting semi-discretized ordinary differential equation (ODE) is multiplying by a test function (shape function) w(t) and integrating over the domain (single element).

$$\int_0^T w(t)\frac{du(t)}{dt}dt = \int_0^T w(t)L(u(x))dt \qquad (2.3)$$

Equation (2.3) is the strong form finite element method. In addition, equation (2.3) can be integrated by parts to obtain DG method form.

$$-\int_0^T \frac{dw}{dt}f(t)dt + \oint_0^T w(t)u_b(t)d\Omega = \int_0^T w(t)L(u)dt. \qquad (2.4)$$

Where f(t) = u(t) is the flux over the element and u_b is flux at the boundaries of the element.

After solving equation (2.3), or equation (2.4). The resultant is an explicit Runge–Kutta method which is commonly written in the Butcher or Shu-Osher forms. Where Butcher form can be written as,

$$\begin{aligned} u^0 &= u^n \\ u^i &= u^n + \Delta T\sum_{j=1}^{i-1} a_{ij}L(u^j), \qquad 1\leq i \leq s \\ u^{n+1} &= u^n + \Delta T\sum_{j=1}^{s} b_{ij}L(u^j) \end{aligned} \qquad (2.5.1)$$

Also Shu-Osher form:

$$\begin{aligned} u^0 &= u^n \\ u^i &= \sum_{j=0}^{i-1} \alpha_{ij}u^j + \beta_{ij}\Delta TL(u^j) \qquad 1\leq i \leq s. \\ u^{n+1} &= u^s, \end{aligned} \qquad (2.5.2)$$

Where, consistency requires that $\sum_{j=0}^{j=i-1} \alpha_{ij} = 1$

However, it was proven by Gottlieb & Shu (1988) and Gottlieb, Ketchson & Shu (2011) that no fourth order-four stages with positive SSP coefficient, where β_{ij} must have at least one negative coefficient. That proven may attribute of using finite difference or standard Galerkin finite elements (*in which the state variable and the weight function are of the same polynomial)*. In this work, we let the distribution points of the state variable is different from the distribution points of the weight function, then the Runge-Kutta of fourth order-four stages can exist and the

L_1 norm of the governing equation will be improved. The four stages fourth order Runge-Kutta RK(4,4) can be written as:

$$
\begin{aligned}
u^0 &= u^n \\
u^1 &= u^0 + 0.4189\,\Delta t L(u^0) \\
u^2 &= [0\,.94522875653802296u^0 + 0.05477124346197704\,u^1 \\
&\quad + 0.101969806730994\ \ \Delta t L(u^0) + 0.4561938275384\,\Delta t L(u^1)\,] \\
u^3 &= u^0 + 0.25\Delta t L(u^0) + 0.75\Delta t L(u^2) \\
u^{n+1} &= u^0 + \Delta t([2.3261848461[\{L(u^0) + L(u^3)] + 1.6738151539[L(u^1) + L(u^2)]])/8.
\end{aligned}
\tag{2.6}
$$

With CFL coefficient of ≈ 1.1 and effective SSP =0.275.

(RK(4,3) is presented in Ruuth (2006) as:

$$
\begin{aligned}
u^0 &= u^n \\
u^1 &= u^0 + 0.5\Delta t L(u^0) \\
u^2 &= u^1 + \Delta t L(u^1) \\
u^3 &= \frac{2}{3}u^0 + \frac{1}{3}(u^2) + \frac{1}{6}\Delta t L(u^2) \\
u^{n+1} &= u^3 + \tfrac{1}{2}\Delta t L(u^2)
\end{aligned}
\tag{2.7}
$$

With CFL coefficient of 2.0 and effective SSP =0.5

RK(5,4) is presented by Ruuth (2006) as:

$$
\begin{aligned}
u^0 &= u^n \\
u^1 &= u^0 + .39175222657189\Delta t L(u^0) \\
u^2 &= .444370493651235u^0 + .555629506348765u^1 + .368410593050371\Delta t L(u^1) \\
u^3 &= \ .620101851488403u^0 + .379898148511597(u^2) \\
&\quad + .251891774271694\Delta t L(u^2) \\
u^4 &= .178079954393132u^0 + .821920045606868u^3 + .544974750228521\Delta t L(u^3) \\
u^{n+1} &= .517231671970585u^2 + 0.096059710526147u^3 \\
&\quad + 0.06369246866629\Delta t L(u^3) + .386708617503269u^4 \\
&\quad + .226007483236906\Delta t L(u^4)
\end{aligned}
\tag{2.8}
$$

This method has SSP coefficient C = 1.508 and effective SSP=0.302

3. Numerical Results.

The numerical experiments are performed to demonstrate the performance and to verify the order of accuracy of the numerical scheme. In this work, the tests examples are used to investigate the L_1 norm error and the order of the accuracy of the scheme by using only polynomial of order k = 3.0 for the one as well as two-dimensional governing equations.

Throughout this work, the global error is calculated as the difference between the exact solutions and the numerical solutions. The discretize L_1 norm errors is given as

$$L_1 = \sum_{j=1}^{j=N} \sum_{i=1}^{i=edof} |(u_i - u_{exc})/NDOF|$$

Where, edof is the element degree of freedom, N is equal to the number of elements and NDOF=N*edof is the total number of degrees of freedom over the problem domain. The order of accuracy of the scheme can be calculated by using the following formula (Wolkov & Leonardo, 2009).

$$Order = \frac{\log\left(\frac{error_2}{error_1}\right)}{\log\left(\frac{NDOF_2}{NDOF_1}\right)}$$

where $error_1$ and $error_2$ are error norms representing the coarser and the finer grid, respectively.

The first test example is one-Dimensional Linear Advection Equation. The governing equation can be written as,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0\,, \qquad \text{with } f(u) = u$$

The initial condition is given as:

$u = \sin(\pi x)$. The exact solution is $u = \sin(\pi(x - t))$

The problem domain [-1, 1] is divided into N equal elements. The shape functions are constructed from the polynomials of orders k =3. The RK (4, 3), RK (4, 4) and RK (5, 4) are used for evaluating the time integral part. The numerical results are obtained at time t = 2. For one-dimensional case, the maximum Courant number is 10./44.=.2272727,which is maximum

CFL for RK(4,4). Table 1. Shows that RK(4,3) has higher errors as compared with RK(4,4) and RK(5,4). In addition, the order of accuracy of RK(5,4) decays for finer grids as compared with RK(4,4). However RK(5,4) works at CFL lower than maximum CFL of the method.

The second test example is two-Dimensional Linear Advection Equation. The governing equation can be written as,

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}+\frac{\partial g(u)}{\partial y}=0 \qquad 0\leq x \leq 4,\quad 0\leq y \leq 4.$$

Where $f(u) = g(u) = u$ with the periodic boundary conditions and the initial condition given as:

$$u=\sin(\frac{\pi}{2}(x+y))$$

The exact solution is the initial solution shifted with 2t as follows

$$u=\sin(\frac{\pi}{2}(x+y-2t)).$$

This test example is used to verify the order of accuracy of the scheme using RK(4,4) and RK(5,4) methods with equally spaced polynomial of order k = 3. The numerical results are obtained at the final time t = 1.0 . Table 2. exhibits the L_1 errors and the accuracy order of the scheme at CFL = 10./48.=0.208333333.

Table 1: The L_1error and the order of accuracy for 1D linear advection equation with periodic boundary conditions at t = 2. by using polynomial of order k = 3.

	RK(4,3)		RK(4,4)		RK(4,5)	
N	L_1	order	L_1	order	L_1	order
10	4.9526550e-05	-	4.216373802e-05	-	4.2162683e-05	-
20	4.71223556e-06	3.394	2.6046661736e-06	4.0168	2.6038335e-06	4.0173
40	5.30970693e-07	3.150	1.624850789e-07	4.0027	1.6245599e-07	4.003
80	6.28021882e-08	3.080	1.015092510e-08	4.0006	1.0148699e-08	4.0007
160	7.6278474e-09	3.041	6.3424038089e-10	4.0004	6.3423518e-10	4.0001
320	9.3958224e-10	3.021	3.965362813e-11	3.9995	3.9638927e-11	4.0000
640	-	-	2.5223819695e-12	3.8616	3.8829359e-12	3.3517

Table 2: The L_1 errors and order of accuracy for the 2D linear advection equation with the periodic boundary conditions at t = 1 using the polynomial of order k = 3

	RK(4,4)		RK(4,5)	
N	L_1	order	L_1	order
10×10	7.13773924080e-05	-	7.1408728814e-05	
20×20	3.97030907001e-06	4.1681	3.9704802277e-06	4.1687
30×30	8.01022621838e-07	3.9478	8.0110367733e-07	3.9477
40×40	2.547149561803e-07	3.9827	2.5473154663e-07	3.9828
50×50	1.0365920857454e-07	4.0290	1.0366357255e-07	4.0291
60×60	5.0101792417526e-08	3.9877	5.0103294098e-08	3.9878
70×70	2.7032435640863e-08	4.0027	2.7033201905e-08	4.0027
80×80	1.584345107535e-08	4.0012	1.5843786658e-08	4.0012
160×160	9.9020930871650e-10	4.000	9.9020825521e-10	4.000

4. Conclusion.

This work presented SSPRK(4,4) and compared it with SSPRK(5,4). The numerical results prove that RK(4,4) does exist and is very efficient as compared with RK(5,4) method. It was also found that the effective SSP of RK(4,4) is lower than that of RK(5,4).

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